On the Growth of Path Algebras

A Review of Gel'fand-Kirillov Dimension and Entropy

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Joint work...

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Overview

Path Algebra

Let $E = (E^0, E^1, s, r)$ be a directed graph with

- *vertices* E^0,
- *edges* E^1 and
- $r, s : E^1 \to E^0$.

For $e \in E^1$, $s(e)$ and $r(e)$ is the *source* and the *range* of e.

$$s(e) \bullet \longrightarrow \bullet\ r(e)$$

Definition

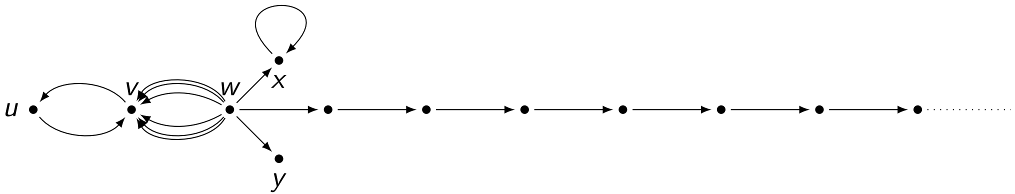
The path algebra KE is the K-algebra with basis $\{p_i\}$ consisting of directed paths in E. (vertices are paths of lengths 0). We have

(V) $v_i v_j = \delta_{i,j} v_i$ *for every* $v_i, v_j \in E^0$;

(E) $s(e)e = e = er(e)$ *for all non-sinks* $e \in E^1$.

Path Algebra

This means one could have



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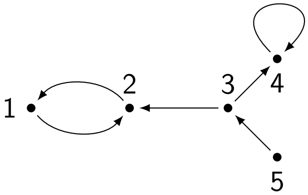
Some notation

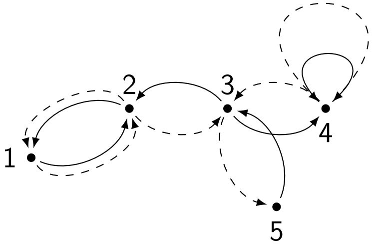
- E is a *finite graph* if $|E^0 \cup E^1| < \infty$.
- A vertex v for which $s^{-1}(v) = \emptyset$ is called a *sink*, a vertex v for which $r^{-1}(v) = \emptyset$ is called a *source*. We will denote the set of sinks of E by $\mathrm{Sink}(E)$ and the set of sources by $\mathrm{Source}(E)$.
- A vertex $v \in E^0$ is a *infinite emitter* if $|s^{-1}(v)| = \infty$.

Definition

We say that E satisfies *Condition (EXC)* if every cycle of E is an exclusive cycle. In other words, a graph with Condition (EXC) is one **without non-disjoint cycles.** A *chain of cycles* of *length* n is a sequence of cycles $C_1, C_2, \cdots, C_n$ such that there is a path from C_i to C_{i+1} for each $i < n$. This chain has an *exit* if the last cycle C_n has an exit.

Adding ghost edges





Double Graph & Leavitt Path Algebra

Let $\hat{E} = (E^0, E^1, {E^1}^* s, r)$ the double graph of E with the set of *ghost edges* ${E^1}^*$, i.e. for $e \in E^1$ we have $e^* \in {E^1}^*$ with

$$s(e^*) = r(e) \text{ and } r(e^*) = s(e).$$

Definition ('05, Abrams& Pino and Ara, Moreno & Pardo)

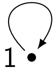
For a graph E and a field K, the *Leavitt path algebra* of E, denoted by $L_K(E)$, is the path algebra over the double graph $\hat{E}$ with additional relations

(CK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$;

(CK2) $\sum_{\{e \in E^1 :\, s(e) = v\}} ee^* = v$ for every v which is not a sink or an infinite emitter.

Fundamental Examples: 1-petal rose & line A_n, see e.g. Abrams, Ara, Sines-Molina, 2017

1-petal rose

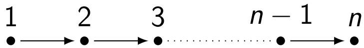


Path algebra: $KR_1 = K[x]$
polynomial algebra with coefficients in K

LPA: $L_K(R_1) = K[x, x^{-1}]$,
Laurent polynomial algebra

Infinite dimensional!

A_n



Path algebra: $KA_n = T_n(K)$,
upper triangular matrix algebra over K

LPA: $L_K(A_n) = M_n(K)$,
matrix algebra over K.

Finite dimensional!

In particular

Theorem

The Leavitt path algebra $L_K(E)$ is a finite dimensional K- algebra if and only if E is a finite and acyclic graph.

Awesome! BUT:

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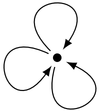
How to distinguish infinite dimensional from infinite dimensional! For example:



Definition (e.g. Abrams, Pino '05)

Let E be a graph and K any field. For $v \in E^0$ and $e \in E^1$, consider $\deg(v) = 0, \deg(e) = 1, \deg(e^*) = -1$. For any nonzero monomial $kx_1 \cdots x_m$ with $k \in K^\times$ and $x_i \in E^0 \cup E^1 \cup (E^1)^*$, define

$$\deg(kx_1 \cdots x_m) = \sum_{i=1}^{m} \deg(x_i).$$

Then for $n \in \mathbb{Z}$ we set

$$A_n := \operatorname{span}\{x_1 \cdots x_m \,|\, x_i \in E^0 \cup E^1 \cup (E^1)^* \text{ with } \deg(x_1 \cdots x_m) = n\}.$$

Natural Grading

Theorem (e.g. Abrams, Pino '05)

Let E be a graph. We have the following $\mathbb{Z}-$gradings on the corresponding path algebras:

i) $K\hat{E} = \bigoplus_{n\in\mathbb{Z}} A_n$ *as K-subspaces*

ii) $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_n$, *where*

$$L_n := \mathrm{span}\{\lambda\mu^* \mid \lambda, \mu \text{ paths and } l(\lambda) - l(\mu) = n\}$$

iii) $KE = \bigoplus_{n\in\mathbb{Z}} K_n$ *with*

$$K_n := \mathrm{span}\{x_1 \cdots x_m \mid x_i \in E^0 \cup E^1 \text{ with } \deg(x_1 \cdots x_m) = n\}$$

Gel'fand-Kirillov Dimension/Polynomial Growth

Definition (Izrail M. Gelfand and Alexander A. Kirillov,'66)

Let A be an algebra generated by a finite dimensional subspace V. Let $V^n = \mathrm{span}\{v_1 v_2 \cdots v_k \mid v_i \in V, k \leq n\}$. Then $V = V^1 \subseteq V^2 \subseteq \cdots$,

$$A = \bigcup_{n\geq 1} V^n \quad \text{and} \quad g_V(n) := \dim V^n < \infty.$$

If W is another finite-dimensional subspace that generates A, then $g_V(n) \sim g_W(n)$. If $g_V(n)$ is polynomially bounded, then the *Gelfand-Kirillov* dimension of A is defined as

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log g_V(n)}{\log(n)}.$$

The GK-dimension does not depend on a choice of the generating space V as long as $\dim(V) < \infty$. If the growth of A is not polynomially bounded, then $\mathrm{GKdim}(A) = \infty$.

Alhamadi, Zelmanov et.al '12

Theorem

Let E be a finite graph.

(1) *The Leavitt path algebra $L_K(E)$ has polynomially bounded growth if and only if E is a graph with disjoint cycles.*

(2) *If d_1 is the maximal length of a chain of cycles in E, and d_2 is the maximal length of chain of cycles with an exit, then*

$$\mathrm{GKdim}(L_K(E)) = \max(2d_1 - 1, 2d_2).$$

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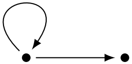
$$\mathrm{GKdim}(L_K(E)) = \max(2d_1 - 1, 2d_2).$$

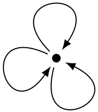
If $g_V(n) = n^k$, then $\mathrm{GKdim}(A) = k$.
If $\dim(A) < \infty$ we have $\mathrm{GKdim}(A) = 0$.

Examples... again

1 cycle, no exit
$\mathrm{GKdim}(L_K(R_1)) = 1$



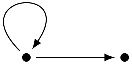
1 cycle with exit
$\mathrm{GKdim}(L_K(E)) = 2$



cycles not disjoint
$\mathrm{GKdim}(L_K(R_3)) = \infty$

Examples... again

1 cycle, no exit
$\mathrm{GKdim}(L_K(R_1)) = 1$



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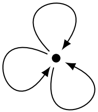
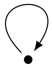


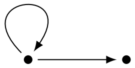
cycles not disjoint
$\mathrm{GKdim}(L_K(R_3)) = \infty$

Better but suboptimal!

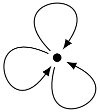
Examples... again



1 cycle, no exit
$\mathrm{GKdim}(L_K(R_1)) = 1$



1 cycle with exit
$\mathrm{GKdim}(L_K(E)) = 2$



cycles not disjoint
$\mathrm{GKdim}(L_K(R_3)) = \infty$

Better but suboptimal!

How to measure exponential growth?

An ignorant Physicist view

Let E be a finite graph. And KE, $L_K(E)$ and $K\hat{E}$ its corresponding path algebras.
From the growth point of view we would expect something like:

$$KE \leq L_K(E) \leq K\hat{E}.$$

Entropy of Filtered Algebras

A K-algebra A is said to be *filtered* if it is endowed with a collection of subspaces $\mathcal{F} = \{V_n\}_{n=0}^{\infty}$ such that

i) $0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots A$,

ii) $A = \cup_{n \geq 0} V_n$,

iii) $V_n V_m \subset V_{n+m}$.

For a graded algebra $A = \oplus_{i=0}^{\infty} A_i$, its entropy has been defined by Newman, 2000, as

$$H(A) = \limsup_{n \to \infty} \sqrt[n]{\dim(A_n)}.$$

Some historical remarks

- Entropy of a thermodynamical system $- >$ Boltzmann (1866)
- algebraic entropy for filtered algebras e.g. Newmann 2000
- algebraic entropy for group (morphisms), e.g. Bruno& Dikranjan '17, Weiss '74
- growth of algebras -> Gel'fand & Kirillov '66, Grigorchuk '83& works of Zelmanov, Kirillov, Gromov,...
- recent works for LPA, Koc et. al. '22, Hazrat, Sebandal, Vilela, '22

Entropy of Filtered Algebras

If$(A, \mathcal{F})$, with filtration $\mathcal{F} = \{V_n\}_{n \geq 0}$ of finite-dimensional quotients V_n/V_{n-1}
Consider the associated graded algebra:

$$\mathbf{gr}(A) := \oplus_{i \geq 0} V_{i+1}/V_i$$

s.t. $(x + V_{n-1})(y + V_{m-1}) := xy + V_{n+m-1}$,
where $x + V_{n-1} \in V_n/V_{n-1}$, $y + V_{m-1} \in V_m/V_{m-1}$.

Then we define the ***algebraic entropy of a filtered algebra*** $(A, \mathcal{F})$ by

$$\mathrm{h}_{\mathrm{alg}}(A, \mathcal{F}) := \begin{cases} 0 & \text{if A is finite dimensional,} \\ \limsup_{n \to \infty} \frac{\log \dim(V_n/V_{n-1})}{n} & \text{otherwise.} \end{cases}$$

Lemma

Let $\mathcal{F} = \{V_n\}$ be a filtration of a finitely generated algebra A. For the filtration $\mathcal{G} = \{W_n\}$ such that $W_n := V_{nk}$ for any $n \in \mathbb{N}$ and a fixed $k \in \mathbb{N}^*$, one has $\mathrm{h}_{\mathrm{alg}}(A, \mathcal{G}) = k \cdot \mathrm{h}_{\mathrm{alg}}(A, \mathcal{F})$.

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Definition

We define the following *standard filtrations:*

i) For KE we define the filtration $\{V_i\}_{i\in\mathbb{N}}$ where V_0 is the linear span of the set of vertices of the graph E, while V_1 is the sum of V_0 with the linear span of the set of edges, and V_{k+1} linear span of the set of paths of length less or equal to $k+1$.

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ii) For $L_K(E)$ we define the filtration $\{W_i\}_{i \in \mathbb{N}}$ so that W_0 is the linear span of the set of vertices of E, being W_1 the sum of W_0 plus the linear span of the set $E^1 \cup (E^1)^*$. For W_k we take the linear span of the set of elements: $\lambda\mu^*$ with $l(\lambda) + l(\mu) \leq k$.

Proposition

Assume that A is a finitely generated algebra and $\mathrm{h}_{\mathrm{alg}}(A, \mathcal{F}) = 0$ for a filtration $\mathcal{F} = \{V_n\}$. Then we have the following.

- *i) For any other filtration $\mathcal{G} = \{W_n\}$ such that $W_n \subset V_n$ for any n, one has $\mathrm{h}_{\mathrm{alg}}(A, \mathcal{G}) = 0$.*
- *ii) For the filtration $\mathcal{G} = \{W_n\}$ such that $W_n := V_{nk}$ for any n and a fixed k, one has $\mathrm{h}_{\mathrm{alg}}(A, \mathcal{G}) = 0$.*
- *iii) For any other filtration $\mathcal{G} = \{W_n\}$ such that W_1 is finite dimensional and $W_k = (W_1)^k$ (for any k), one has $\mathrm{h}_{\mathrm{alg}}(A, \mathcal{G}) = 0$.*

Proposition

Suppose that A is a K-algebra and $\mathcal{F} = \{V_n\}$ a filtration of A with V_1 a finite dimensional system of generators with $(V_1)^n = V_n$.

- *i) If* $\lim\limits_{n\to\infty} \dim(V_n/V_{n-1}) = 0$, *then* $h_{alg}(A) = 0$ *and* $\mathrm{GKdim}(A) = 0$.
- *ii) If* $\lim\limits_{n\to\infty} \dim(V_n/V_{n-1}) = c > 0$, *then* $h_{alg}(A) = 0$ *and* $\mathrm{GKdim}(A) = 1$.
- *iii) If* $\dim(V_n) = \Theta(n^k)$ *for some* $k \in \mathbb{N}^*$, *then* $h_{alg}(A) = 0$, *and* $\mathrm{GKdim}(A) = k$.
- *iv) If* $\dim(V_n) = \Theta(a^n)$, *then* $\mathrm{h}_{\mathrm{alg}}(A) = \log(a)$ *and* $\mathrm{GKdim}(A) = \infty$.

n-petal rose R_n, standard filtration

$$\dim(V_1) - \dim(V_0) = 2n,\ \dim(V_2) - \dim(V_1) = 3n^2 - 1, \ldots$$

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$\dim(V_1) - \dim(V_0) = 2n$, $\dim(V_2) - \dim(V_1) = 3n^2 - 1$,...

gen. of V_k = gen. of V_{k-1} + elements of $(E^1)^k \cup (E^1)^{k*} \cup \left(\cup_{i+j=k}(E^1)^i(E^{1^{j*}})\right)$.

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For a basis we remove from each $(E^1)^i(E^{1*})^j$ the elements $(E^1)^{i-1}f_1 f_1^*(E^{1*})^{j-1}$ (so remove $n^{i+j-2} = n^{k-2}$ elements)

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Hence $\dim(V_k/V_{k-1}) = (k+1)n^k - (k-1)n^{k-2}$ thus

$$\mathrm{h}_{\mathrm{alg}}(L_k(R_n)) = \limsup_{k\to\infty} \frac{\log[(k+1)n^k - (k-1)n^{k-2}]}{k} = \log(n)$$

A very helpful tool

Definition

Let E be a finite directed graph with n vertices. The *adjacency matrix* of E denoted by $A_E := (a_{i,j})_{n\times n}$ where $a_{i,j} = |\{e \in E^1 : s(e) = v_i, r(e) = v_j\}|$.

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Lemma

Let A_E be the adjacency matrix associated to a finite directed graph E. Then

$$\mathrm{h}_{\mathrm{alg}}(KE) = \limsup_{n\to\infty} \frac{\log(\|A_E^n\|_{1,1})}{n},$$

with $\|A_E\|_{1,1} := \sum_{i,j=1}^{m} |a_{i,j}|$.
In particular:

$$\mathrm{h}_{\mathrm{alg}}(KE) = \log(\rho(A_E)), \quad \textit{with } \rho(A_E) \textit{ the spectral radius.}$$

Proof.

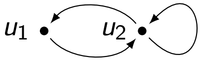
If $A_E^n = (a_{i,j})$, then $a_{i,j}$ is the number of paths of length n from the vertex v_i to the vertex v_j in the graph.

Thus

$$\|A_E^n\|_{1,1} = \sum_{i,j=1}^{m} a_{i,j} = |\{\mu \in \mathrm{Path}(E) \colon l(\mu) = n\}| = \dim\left(V_n / V_{n-1}\right),$$

where $\{V_i\}_{i \geq 0}$ is the standard filtration of KE. □

Fibonacci...



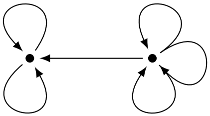
Example

$$A_E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A_E^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix} \quad \text{where } f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ \text{and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3.$$

$$\mathrm{h}_{\mathrm{alg}}(KE) = \limsup_{n\to\infty} \frac{\log(\|A_E^n\|)}{n} = \lim_{n\to\infty} \log\left(\frac{e_+^{n+1} - e_-^{n+1} + e_+^n - e_-^n}{e_+^n - e_-^n + e_+^{n-1} - e_-^{n-1}}\right) = \log(\frac{1+\sqrt{5}}{2}).$$

Unconnected roses



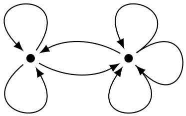
Example

Let us now connect the two roses R_2 and R_3 with 2 and 3 petals with one edge. Then the graph has the adjacency matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

$$h_{alg}(KE) = \log(3) = \log(\max\{2,3\}).$$

Connected roses



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$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

$$h_{alg}(KE) = \log(\frac{5}{2} + \sqrt{\frac{17}{4}}).$$

Lemma

Let C_n be the cycle with n vertices. Then:

$$\mathrm{h}_{\mathrm{alg}}(KC_n) = 0,$$

$$\mathrm{h}_{\mathrm{alg}}(K\hat{C}_n) = \log(2), \text{ and}$$

$$\mathrm{h}_{\mathrm{alg}}(L_K(C_n)) = 0.$$

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$$\mathrm{h_{alg}}(L_K(C_n)) = 0.$$

Sketch.

$$A_{C_n} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

we have $\|A_{C_n}^m\| = m, \forall m$, hence

$$h_{alg}(KC_n) = \limsup_{m\to\infty} \frac{\log(m)}{m} = 0.$$

Lemma

Let C_n be the cycle with n vertices. Then:

$\mathrm{h}_{\mathrm{alg}}(KC_n) = 0,$

$\mathrm{h}_{\mathrm{alg}}(K\hat{C}_n) = \log(2),$ *and*

$\mathrm{h}_{\mathrm{alg}}(L_K(C_n)) = 0.$

Sketch.

$$A_{\hat{C}_n} = \begin{pmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

we have $\rho(A_{\hat{C}_n}) = 2,$, hence

$$h_{alg}(K\hat{C}_n) = \log(2).$$

□

Theorem

For any finite directed graph E, we have that

$$h_{alg}(L_K(E)) = h_{alg}(KE) = h_{alg}(C_K(E)),$$

where the latter denotes the Cohn path algebra.

Lemma

Let E be a finite directed connected graph satisfying Condition (EXC) and without sources and sinks. Let $A \in \{KE, L_K(E), C_K(E)\}$, then $\mathrm{h}_{\mathrm{alg}}(A) = 0$.

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Theorem

Let E be a finite directed graph and $A \in \{KE, L_K(E), C_K(E)\}$, the associated path algebra, then

i) $\mathrm{GKdim}(A) = 0$ *if and only if A is finite-dimensional;*

ii) If $0 \neq \mathrm{GKdim}(A) < \infty$, *then* $\mathrm{h_{alg}}(A) = 0$ *and KE is infinite dimensional.*

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Lemma

Let A be a filtered algebra with filtration $\{V_n\}_{n>0}$ such that $h_{alg}(A) = \infty$, then $\dim(V_n/V_{n-1})$ *grows superexponential, i.e.*

$$\limsup_{n\to\infty} \frac{\dim(V_n/V_{n-1})}{c^n} = \infty, \qquad \text{for any } c > 0.$$

Trichotomy Theorem for Finite Graphs

Theorem (Growth Trichotomy)

Let $A \in \{KE, L_K(E), C_K(E)\}$ *for a finite graph* E*. Then* A *can be classified into three types as follows: if* $t(A, \mathcal{F}) := (\dim(A), \mathrm{GKdim}(A), \mathrm{h}_{\mathrm{alg}}(A))$ *one has*

$$t(A, \mathcal{F}) = (k, 0, 0) \text{ for } k < \infty; \text{ or}$$

$$t(A, \mathcal{F}) = (\infty, l, 0) \text{ for } l < \infty; \text{ or}$$

$$t(A, \mathcal{F}) = (\infty, \infty, m) \text{ for } m < \infty.$$

In particular, we have $h_{alg}(A) < \infty$.

Sketch of proof:

Proof.

To show: $h_{alg}(L_K(E)) < \infty$:
Assume $\exists p > 1$ s.t.

$$\dim(V_n/V_{n-1}) > c^{(n^p)}, \quad \forall n > N$$

Then

$$\dim(V_{n+1}/V_n) = \dim(V_n/V_{n-1}) + \#\text{path of length } n+1 > c^{(n+1)^p}$$

Hence

$$\#\text{path}/c^{(n+1)^p} > 1 - g(V(n))/c^{(n+1)^p} > 0, \quad \forall n.$$

Hence the number of paths grows superexponential!
But $\mathrm{growth}(L_K(E)) \leq K\hat{E} \leq \mathcal{O}(e^{2dn})$, where d is the maximum outbound degree of all vertices. Since the graph is finite d is finite. **Contradiction!** □

Thank you very much!

Tack så mycket!

Daghang salamat!