# Representations of quantum affine Lie algebras

Vyacheslav Futorny

Shenzhen International Center for Mathematics
Southern University of Science and Technology

**Joint results with Xingpeng Liu, Juan Camilo Arias and Kailash Misra**

Linkoping - 2026

## Affine Lie algebras

$\mathfrak{g}$ a simple finite-dimensional Lie algebra

Loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with the Lie bracket
$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n}$

Non-twisted affine Kac-Moody algebra is the universal central extension $\widehat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$,

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n(x, y)\delta_{n+m,0}c,$$

$d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ degree derivation, $d(x \otimes t^n) = n(x \otimes t^n)$, $d(c) = 0$

$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c \oplus \mathbb{C}d$ Cartan subalgebra

## Borel and parabolic subalgebras

$\diamond$ A subset $P \subset \Delta$ is a quase partition if:
a) $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$
b) Let $\mathcal{B}_P$ a Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by $\widehat{\mathfrak{h}}$ and the root spaces $\widehat{\mathfrak{g}}_\alpha$ with $\alpha \in P$. Then for any root $\alpha$ of $\mathcal{B}_P$ we have $\alpha \in P$

$\diamond$ $\mathcal{B}_P$ describes (almost all) Borel subalgebras of $\widehat{\mathfrak{g}}$. There exists a finite number conjugacy classes, (roughly) parameterized by the parabolic subalgebras of $\mathfrak{g}$ (Jacobsen-Kac; VF)

**Example**

- standard Borel subalgebra with the partition $P = \Delta^+$
- natural Borel subalgebra $\mathcal{B}_{\mathrm{nat}}$ with the partition

$$P_{\mathrm{nat}} = \{\alpha + k\delta | \alpha \in \Delta^+(\mathfrak{g}), k \in \mathbb{Z}\} \cup \{n\delta | n \geq 0\}$$

- each function $\phi : \mathbb{N} \to \mathbb{Z}_2$ defines a Borel subalgebra $\mathcal{B}_{\mathrm{nat}}^{\phi}$ with quase partition $\{\alpha + k\delta | \alpha \in \Delta^+, k \in \mathbb{Z}\} \cup$ $\{n\delta \mid n \in \mathbb{N}, \phi(n) = 1\} \cup \{-m\delta \mid m \in \mathbb{N}, \phi(m) = 0\}$

# Closed partitions in $\hat{\mathfrak{sl}}_2$

Standard partition:

$$\begin{array}{cccccccc} \cdots & -\alpha_0-\delta & -\alpha_0 & \alpha_1 & \alpha_1+\delta & \alpha_1+2\delta & \alpha_1+3\delta & \cdots \\ \cdots & -2\delta & -\delta & & \delta & 2\delta & 3\delta & \cdots \\ \cdots & -\alpha_1-2\delta & -\alpha_1-\delta & -\alpha_1 & \alpha_0 & \alpha_0+\delta & \alpha_0+2\delta & \cdots \end{array}$$

# Closed partitions in $\hat{\mathfrak{sl}}_2$

Standard partition:

$$\begin{array}{cccccccc}
\cdots & -\alpha_0-\delta & -\alpha_0 & \alpha_1 & \alpha_1+\delta & \alpha_1+2\delta & \alpha_1+3\delta & \cdots \\
\cdots & -2\delta & -\delta & & \delta & 2\delta & 3\delta & \cdots \\
\cdots & -\alpha_1-2\delta & -\alpha_1-\delta & -\alpha_1 & \alpha_0 & \alpha_0+\delta & \alpha_0+2\delta & \cdots
\end{array}$$

Natural partition:

$$\begin{array}{cccccccc}
\cdots & -\alpha_0-\delta & -\alpha_0 & \alpha_1 & \alpha_1+\delta & \alpha_1+2\delta & \alpha_1+3\delta & \cdots \\
\cdots & -2\delta & -\delta & & \delta & 2\delta & 3\delta & \cdots \\
\cdots & -\alpha_1-2\delta & -\alpha_1-\delta & -\alpha_1 & \alpha_0 & \alpha_0+\delta & \alpha_0+2\delta & \cdots
\end{array}$$

$\diamond$ A **closed** subset $P \subset \Delta$ such that $P \cup (-P) = \Delta$ defines a parabolic subalgebra $\mathcal{P} \subset \widehat{\mathfrak{g}}$

$\diamond$ A parabolic subalgebra $\mathcal{P} \subset \widehat{\mathfrak{g}}$ contains a Borel subalgebra:

a) **type I** (contains the standard Borel):
$\mathcal{P} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l}$ is finite-dimensional reductive Lie algebra;

b) **type II** (contains one of Borel subalgebras $\mathcal{B}^{\phi}_{\mathrm{nat}}$):
$\mathcal{P} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l}$ is infinite-dimensional Lie subalgebra

**Parabolic induction**

$\diamond$ Borel subalgebra B $\rightsquigarrow$ Verma type module

$$M_B(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(B)} \mathbb{C}$$

$\diamond$ Parabolic subalgebra $\mathcal{P}$ $\rightsquigarrow$ generalized Verma type module

$$M_{\mathcal{P}}(N) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathcal{P})} N$$

Theorem (F., Kashuba et al, 2016-2023)

*Let $\mathcal{P} \subset \widehat{\mathfrak{g}}$ be a parabolic subalgebra of type II, $N$ is a nice simple $\mathcal{P}$-module with $c \neq 0$ Then $M_{\mathcal{P}}(N)$ is a simple $\widehat{\mathfrak{g}}$-module*

## Imaginary Verma modules

For the natural closed partition

$$S = \{\alpha + n\delta | \alpha \in \Delta^+(\mathfrak{g}), n \in \mathbb{Z}\} \cup \{k\delta | k > 0\}$$

consider the following subalgebras of $\hat{\mathfrak{g}}$: $\mathfrak{g}_\pm(S) = \sum_{\alpha \in S} \hat{\mathfrak{g}}_{\pm\alpha}$.
For example, for $\mathfrak{g} = sl(2)$ we have

- $\mathfrak{g}_+(S)$ is the subalgebra generated by $e(k) = \begin{pmatrix} 0 & t^k \\ 0 & 0 \end{pmatrix}$, $(k \in \mathbb{Z})$ and $h(l) = \begin{pmatrix} t^l & 0 \\ 0 & -t^l \end{pmatrix}$, $(l \in \mathbb{Z}_{>0})$.
- $\mathfrak{g}_-(S)$ is the subalgebra generated by $f(k) = \begin{pmatrix} 0 & 0 \\ t^k & 0 \end{pmatrix}$, $(k \in \mathbb{Z})$ and $h(-l) = \begin{pmatrix} t^{-l} & 0 \\ 0 & -t^{-l} \end{pmatrix}$, $(l \in \mathbb{Z}_{>0})$,

$$\hat{\mathfrak{g}} = \mathfrak{g}_-(S) \oplus \hat{\mathfrak{h}} \oplus \mathfrak{g}_+(S),$$

and

$$U(\hat{\mathfrak{g}}) \cong U(\mathfrak{g}_-(S)) \otimes U(\hat{\mathfrak{h}}) \otimes U(\mathfrak{g}_+(S))$$

A $U(\hat{\mathfrak{g}})$-module $V$ is called a *weight* module if $V = \bigoplus_{\mu \in \hat{\mathfrak{h}}^*} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, c \cdot v = \mu(c)v, d \cdot v = \mu(d)v\}.$$

A $U(\hat{\mathfrak{g}})$-module $V$ is called an *$S$-highest weight module* with highest weight $\lambda$ if there is a non-zero $v_\lambda \in V$ such that

- $u^+ \cdot v_\lambda = 0$ for all $u^+ \in \mathfrak{g}_+(S) \setminus \mathbb{C}^*$,
- $h \cdot v_\lambda = \lambda(h)v_\lambda$, $c \cdot v_\lambda = \lambda(c)v_\lambda$, $d \cdot v_\lambda = \lambda(d)v_\lambda$,
- $V = U(\hat{\mathfrak{g}}) \cdot v_\lambda = U(\mathfrak{g}_-(S)) \cdot v_\lambda$.

For $\lambda \in \hat{\mathfrak{h}}^*$, let $I_S(\lambda)$ denote the ideal of $U(\widehat{\mathfrak{sl}(2)})$ generated by $e(k)$ ($k \in \mathbb{Z}$), $h(l)$ ($l > 0$), $h - \lambda(h)1$, $c - \lambda(c)1$, $d - \lambda(d)1$. Then the *imaginary Verma module* of $\widehat{\mathfrak{sl}(2)}$ with highest weight $\lambda$ is defined as

$$M_S(\lambda) = U(\widehat{\mathfrak{sl}(2)})/I_S(\lambda)$$

## Theorem

(i) $M_S(\lambda)$ is a $U(\mathfrak{g}_-(S))$-free module of rank 1 generated by the $S$-highest weight vector $1 \otimes 1$ of weight $\lambda$.
(ii) $\dim M_S(\lambda)_\lambda = 1$; $0 < \dim M_S(\lambda)_{\lambda - k\delta} < \infty$ for any integer $k > 0$; if $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and $M_S(\lambda)_\mu \neq 0$, then $\dim M(\lambda)_\mu = \infty$.
(iii) $M_S(\lambda)$ has a unique maximal submodule.
(iv) $M_S(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.
(v) Let $\lambda(c) = 0$ then $(U(\sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{\mathfrak{g}}_{k\delta} \oplus \mathbb{C}c) \setminus \{\mathbb{C}\}) \cdot 1$ is a proper submodule of $M_S(\lambda)$. Moreover, it is maximal if and only of $\lambda(h) \neq 0$

$$
\begin{tikzcd}
 & \mathrm{U}(\mathfrak{g}) \arrow[dd, shift left=2, "q\text{−deformation}"] \\
\mathfrak{g} \arrow[ur, squiggly] \arrow[dr, squiggly] & \\
 & \mathrm{U}_q(\mathfrak{g}) \arrow[uu, shift left=2, "q \to 1"]
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & \mathrm{U}(\mathfrak{g}) \arrow[d, shift left, "q-\text{deformation}"] \\
\mathfrak{g} \arrow[urr, rightsquigarrow] \arrow[drr, rightsquigarrow] & & \\
 & & \mathrm{U}_q(\mathfrak{g}) \arrow[u, shift left, "q\to 1"]
\end{tikzcd}$$

Let $\mathrm{Rep}\mathrm{U}_q(\mathfrak{g})$ be the category of finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-modules

$$\begin{tikzcd}
\mathrm{Rep}\mathrm{U}_{q\to 1}(\mathfrak{g}) & & \mathrm{Rep}\mathrm{U}_{q\to 1}(\mathfrak{g}) \\
 & \boldsymbol{\mathrm{Rep}\mathrm{U}_q(\mathfrak{g})} \arrow[ul, "\boldsymbol{q\to 1}"', "="] \arrow[ur, "\text{generic } q", "\cong"'] \arrow[dl, "\boldsymbol{q\to 0}"'] \arrow[dr, "q=\epsilon \text{ root of unity}"] & \\
\text{Kashiwara's Crystal Basis} & & \mathrm{Rep}\mathrm{U}_\epsilon(\mathfrak{g})
\end{tikzcd}$$

• Recall that a simple weight Harish-Chandra $\mathfrak{g}$-module $V$ ($\dim V_\lambda < \infty$ for any $\lambda \in \mathfrak{h}^*$) which cannot be induced from a module over any proper parabolic subalgebra of $\mathfrak{g}$ is called *cuspidal*. It is known that such modules exist only when $\mathfrak{g}$ consists of simple components of type $A$ or type $C$.

### Theorem (F–Tsylke, 2001)

*Let $V$ be a simple Harish-Chandra $\hat{\mathfrak{g}}$-module with a nonzero action of $c$. Then $V \simeq L_{k,\mathfrak{p}}(N)$, for some parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ and a simple cuspidal $\mathfrak{l}$-module $N \simeq \otimes D_{S_i}^{\mu_i} L(\lambda_i)$ for some sets of roots $S_i$'s, $\mu_i \in \mathfrak{h}^*$ and simple $\lambda_i$-highest weight $\mathfrak{l}_i$-modules $L(\lambda_i)$, where $\mathfrak{l}_i$'s are simple components of $\mathfrak{l}$ of type $A$ and $C$.*

- Let $\check{U}_q := U_q \otimes_{\mathbb{C}} \Bbbk$, where $\Bbbk$ is the algebraic closure of $\mathbb{C}(q)$.

**Theorem (V.F., X.Liu, 2025)**

*The algebra $\check{U}_q$ admits cuspidal modules if and only if the underlying semisimple Lie algebra $\mathfrak{g}$ consists of simple components of type A, B or C (in contrast, for $U_{\mathbb{Q}(q)}$ cuspidal modules of type B do not exist).*

**Remark:** The result holds not only for an indeterminate q but also for any specialization $q \longrightarrow \xi$, where $\xi \in \mathbb{C}$ is transcendental.

## Example (X.Liu)

Consider the quantum group $\mathrm{U}_q(\mathfrak{g})$ of type $B_2 (= C_2)$ with the Cartan matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

and generators $E_i, F_i, K_i^{\pm 1}, i = 1, 2$. Set $q_1 = q^2, q_2 = q$. Define $V$ as the vector space spanned by $|\boldsymbol{m_1}, \boldsymbol{m_2}\rangle$ for $m_1, m_2 \in \mathbb{Z}_{\geq 0}$. There exists a $\mathrm{U}_q(\mathfrak{g})$-module structure on $V$:

$$\begin{aligned}
\boldsymbol{K_1 . |m_1, m_2\rangle} &= \boldsymbol{q_1^{-m_1+m_2} |m_1, m_2\rangle}, & \boldsymbol{K_2 . |m_1, m_2\rangle} &= \boldsymbol{\sqrt{-1} q_2^{-2m_2-1} |m_1, m_2\rangle}, \\
E_1 . |m_1, m_2\rangle &= [m_1]_{q_1} |m_1 - 1, m_2 + 1\rangle, & E_2 . |m_1, m_2\rangle &= \sqrt{-1} \frac{q_2 + q_2}{q_2 - q_2} [m_2]_{q_1} |m_1, m_2 - 1\rangle, \\
F_1 . |m_1, m_2\rangle &= [m_2]_{q_1} |m_1 + 1, m_2 - 1\rangle, & F_2 . |m_1, m_2\rangle &= |m_1, m_2 + 1\rangle.
\end{aligned}$$

## Example (X.Liu)

Consider the quantum group $\mathrm{U}_q(\mathfrak{g})$ of type $B_2(= C_2)$ with the Cartan matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

and generators $E_i, F_i, K_i^{\pm 1}, i = 1, 2$. Set $q_1 = q^2, q_2 = q$. Define $V$ as the vector space spanned by $|\boldsymbol{m_1}, \boldsymbol{m_2}\rangle$ for $m_1, m_2 \in \mathbb{Z}_{\geq 0}$. There exists a $\mathrm{U}_q(\mathfrak{g})$-module structure on $V$:

$$\begin{aligned}
K_1.|m_1, m_2\rangle &= q_1^{-m_1+m_2}|m_1, m_2\rangle, & K_2.|m_1, m_2\rangle &= \sqrt{-1}q_2^{-2m_2-1}|m_1, m_2\rangle, \\
E_1.|m_1, m_2\rangle &= [m_1]_{q_1}|m_1 - 1, m_2 + 1\rangle, & E_2.|m_1, m_2\rangle &= \sqrt{-1}\frac{q_2 + q_2}{q_2 - q_2}[m_2]_{q_1}|m_1, m_2 - 1\rangle, \\
F_1.|m_1, m_2\rangle &= [m_2]_{q_1}|m_1 + 1, m_2 - 1\rangle, & F_2.|m_1, m_2\rangle &= |m_1, m_2 + 1\rangle.
\end{aligned}$$

This module has a nice property that it is infinite dimensional, the "minimal" bounded weight module for $\mathrm{U}_q(\mathfrak{g})$.

## The quantum group $U_q(\hat{\mathfrak{g}})$

We denote by $U_q(\hat{\mathfrak{g}})$ the quantum affine algebra associated to $\hat{\mathfrak{g}}$: the associative unital $\mathbb{C}(q^{1/2})$-algebra with generators $E_i, F_i, K_\alpha, \gamma^{\pm 1/2}, D^{\pm 1}$ for $0 \leq i \leq N$, $\alpha \in Q$ and defining relations:

$$DD^{-1} = D^{-1}D = K_\alpha K_{-\alpha} = K_{-\alpha}K_\alpha = \gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1$$

$$[\gamma^{\pm 1/2}, U_q(\hat{\mathfrak{g}})] = [D, K_{\pm\alpha}] = [K_\alpha, K_\beta] = 0$$

$$(\gamma^{\pm 1/2})^2 = K_{\pm\delta}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$K_\alpha E_i K_{-\alpha} = q^{(\alpha|\alpha_i)}E_i, \quad K_\alpha F_i K_{-\alpha} = q^{-(\alpha|\alpha_i)}F_i$$

$$DE_iD^{-1} = q^{\delta_{i,0}}E_i, \quad DF_iD^{-1} = q^{-\delta_{i,0}}F_i$$

$$\sum_{s=0}^{1-a_{ij}}(-1)^s E_i^{(1-a_{ij}-s)}E_jE_i^{(s)} = \sum_{s=0}^{1-a_{ij}}(-1)^s F_i^{(1-a_{ij}-s)}F_jF_i^{(s)} = 0, \quad i \neq j$$

where $q_i = q^{d_i}$, $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[n]_i! = [n]_i[n-1]_i \cdots [2]_i[1]_i$, $K_i = K_{\alpha_i}$, $E_i^{(s)} = E_i^s/[s]_i!$ and $F_i^{(s)} = F_i^s/[s]_i!$

Similar to the loop space realization of $\hat{\mathfrak{g}}$ we have the Drinfel'd realization: the generators are $x_{ir}^{\pm 1}, h_{is}, K_i^{\pm 1}, \gamma^{\pm 1/2}, D^{\pm 1}$ for $i \in I_0 = \{1, \dots, n\}$, $r, s \in \mathbb{Z}$ and $s \neq 0$ subject to the relations:

$$D^{\pm 1}D^{\mp 1} = K_i^{\pm 1}K_i^{\mp 1} = \gamma^{\pm 1/2}\gamma^{\mp 1/2} = 1$$
$$[\gamma^{\pm 1/2}, U_q(\hat{\mathfrak{g}})] = [D, K_i^{\pm 1}] = [K_i, K_j] = [K_i, h_{js}] = 0$$
$$Dh_{ir}D^{-1} = q^r h_{ir}, \qquad Dx_{ir}^{\pm}D^{-1} = q^r x_{ir}^{\pm}$$
$$K_i x_{jr}^{\pm} K_i^{-1} = q^{\pm(\alpha_i|\alpha_j)} x_{jr}^{\pm}$$
$$[h_{ik}, h_{jl}] = \delta_{k,-l}\frac{1}{k}[ka_{ij}]_i \frac{\gamma^k - \gamma^{-k}}{q_j - q_j^{-1}}$$
$$[h_{ik}, x_{jl}^{\pm}] = \pm\frac{1}{k}[ka_{ij}]_i \gamma^{\mp|k|/2} x_{j,k+l}^{\pm}$$
$$x_{i,k+1}^{\pm}x_{jl}^{\pm} - q^{\pm(\alpha_i|\alpha_j)}x_{jl}^{\pm}x_{i,k+1}^{\pm} = q^{\pm(\alpha_i|\alpha_j)}x_{ik}^{\pm}x_{j,l+1}^{\pm} - x_{j,l+1}^{\pm}x_{ik}^{\pm}$$
$$[x_{ik}^{+}, x_{jl}^{-}] = \delta_{ij}\frac{1}{q_i - q_i^{-1}}(\gamma^{(k-l)/2}\psi_{i,k+l} - \gamma^{(l-k)/2}\phi_{i,k+l})$$

where

$$\sum_{k=0}^{\infty}\psi_{ik}z^k = K_i \exp\Big((q_i - q_i^{-1})\sum_{l>0} h_{il}z^l\Big)$$
$$\sum_{k=0}^{\infty}\phi_{i,-k}z^{-k} = K_i^{-1} \exp\Big(-(q_i - q_i^{-1})\sum_{l>0} h_{i,-l}z^{-l}\Big)$$

and for $i \neq j$,

$$\mathrm{Sym}_{k_1,\ldots,k_{1-a_{ij}}} \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i x^{\pm}_{ik_1} \cdots x^{\pm}_{ik_r} x^{\pm}_{ij} x^{\pm}_{ik_{r+1}} \cdots x^{\pm}_{ik_{1-a_{ij}}} = 0.$$

If we consider the following generating functions

$$\phi_i(u) = \sum_{p\in\mathbb{Z}} \phi_{ip} u^{-p}, \quad \psi_i(u) = \sum_{p\in\mathbb{Z}} \psi_{ip} u^{-p}, \quad x^{\pm}_i(u) = \sum_{p\in\mathbb{Z}} x^{\pm}_{ip} u^{-p}$$

the defining relations become:

$$\begin{gathered}
[\phi_i(u), \phi_j(v)] = [\psi_i(u), \psi_j(v)] = 0 \\
\phi_i(u)\psi_j(v)\phi_i(u)^{-1}\psi_j(v)^{-1} = g_{ij}(uv^{-1}\gamma^1)/g_{ij}(uv^{-1}\gamma) \\
\phi_i(u)x^{\pm}_j(v)\phi_i(u)^{-1} = g_{ij}(uv^{-1}\gamma^{\mp 1/2})^{\pm 1} x^{\pm}_j(v) \\
\psi_i(u)x^{\pm}_j(v)\psi_i(u)^{-1} = g_{ji}(vu^{-1}\gamma^{\mp 1/2})^{\mp 1} x^{\pm}_j(v) \\
(u - q^{\pm(\alpha_i|\alpha_j)v})x^{\pm}_i(u)x^{\pm}_j(v) = (q^{\pm(\alpha_i|\alpha_j)u-v})x^{\pm}_j(v)x^{\pm}_i(u) \\
[x^+_i(u), x^-_j(v)] = \delta_{ij}(q_i - q_i^{-1})(\delta(u/v\gamma)\psi_i(v\gamma^{1/2}) - \delta(u\gamma/v)\phi_i(u\gamma^{1/2}))
\end{gathered}$$

where $g_{ij}(t) = g_{ij,q}(t)$ is the Taylor expansion at $t = 0$ of the function $(q^{(\alpha_i|\alpha_j)}t - 1)/(t - q^{(\alpha_i|\alpha_j)})$ and $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$.

**Conjecture** [V.F., X.Liu, 2026]:

Let $V_q$ be a simple Harish-Chandra $U_q(\hat{\mathfrak{g}})$-module with a nonzero action of $c$. Then $V_q \simeq L_{k,\mathfrak{p}}(N_q)$, for some parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ and a simple cuspidal $U_q(\mathfrak{l})$-module $N \simeq \otimes D_{S_i}^{\mu_i} L_q(\lambda_i)$ for some sets of roots $S_i$'s, $\mu_i \in \mathfrak{h}^*$ and simple $\lambda_i$-highest weight $\mathfrak{l}_i$-modules $L_q(\lambda_i)$, where $\mathfrak{l}_i$'s are simple components of $\mathfrak{l}$ of type $A$, $B$ and $C$.

## Quantum imaginary Verma modules

For the natural closed partition

$$S = \{\alpha + n\delta | \alpha \in \Delta^{+}(\mathfrak{g}), n \in \mathbb{Z}\} \cup \{k\delta | k > 0\}$$

consider the following subalgebras of $U_q(\hat{\mathfrak{g}})$:

- $U_q^{+}(S)$ generated by $x_{ik}^{+}, h_{il}$ for $i \in I_0$, $k \in \mathbb{Z}$ and $l > 0$.
- $U_q^{-}(S)$ generated by $x_{ik}^{-}, h_{i,-l}$ for $i \in I_0$, $k \in \mathbb{Z}$ and $l > 0$.
- $U_q^{0}(S)$ generated by $K_i^{\pm 1}, \gamma^{\pm 1/2}, D^{\pm 1}$ for $i \in I_0$.

Let $P$ be the integral weight lattice of $\hat{\mathfrak{g}}$, $\lambda \in P$. A weight module $V$ of $U_q(\hat{\mathfrak{g}})$ is called an $S$-highest weight module with highest weight $\lambda$ if there is a non zero vector $v \in V$ of weight $\lambda$ such that $u^{+} v = 0$ for all $u^{+} \in U_q^{+}(S) \setminus \mathbb{C}(q^{1/2})$ and $V = U_q(\hat{\mathfrak{g}}) v$

Consider the Borel subalgebra $B_q$ of $U_q(\hat{\mathfrak{g}})$ generated by $U_q^+(S) \cup U_q^0(\hat{\mathfrak{g}})$, and a one dimensional $B_q$-module $\mathbb{C}(q^{1/2})_\lambda$ with a generator $\mathbf{1}$, on which $U_q^+(S)$ acts trivially and $K_i^{\pm 1}\mathbf{1} = q^{\pm\lambda(h_i)}\mathbf{1}$, $i \in I_0$, $\gamma^{\pm 1/2}\mathbf{1} = q^{\pm\lambda(c)/2}\mathbf{1}$ and $D^{\pm 1}\mathbf{1} = q^{\pm\lambda(d)}\mathbf{1}$.

The **imaginary Verma module** $M_q(\lambda)$ of weight $\lambda \in P$ is defined as

$$M_q(\lambda) = M_{q,S}(\lambda) := U_q(\hat{\mathfrak{g}}) \otimes_{B_q} \mathbb{C}(q^{1/2})_\lambda$$

Consider the Borel subalgebra $B_q$ of $U_q(\hat{\mathfrak{g}})$ generated by $U_q^+(S) \cup U_q^0(\hat{\mathfrak{g}})$, and a one dimensional $B_q$-module $\mathbb{C}(q^{1/2})_\lambda$ with a generator $\mathbf{1}$, on which $U_q^+(S)$ acts trivially and $K_i^{\pm 1}\mathbf{1} = q^{\pm\lambda(h_i)}\mathbf{1}$, $i \in I_0$, $\gamma^{\pm 1/2}\mathbf{1} = q^{\pm\lambda(c)/2}\mathbf{1}$ and $D^{\pm 1}\mathbf{1} = q^{\pm\lambda(d)}\mathbf{1}$.

The **imaginary Verma module** $M_q(\lambda)$ of weight $\lambda \in P$ is defined as

$$M_q(\lambda) = M_{q,S}(\lambda) := U_q(\hat{\mathfrak{g}}) \otimes_{B_q} \mathbb{C}(q^{1/2})_\lambda$$

Theorem [F., Grishkov, Melville, 2005]

$M_q(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.

• Let $\lambda(c) = 0$. Denote by $J^q(\lambda)$ the left ideal of $U_q(\hat{\mathfrak{g}})$ generated by $x_{ik}^+, h_{il}$ for $i \in I_0, k, l \in \mathbb{Z}, l \neq 0$ and $K_i^{\pm 1} - q^{\pm\lambda(h_i)}$, $\gamma^{\pm 1/2} - 1$ and $D^{\pm 1} - q^{\pm\lambda(d)}$. Set

$$\tilde{M}_q(\lambda) = U_q(\hat{\mathfrak{g}})/J^q(\lambda)$$

It is a quotient of $M_q(\lambda)$, we call it **reduced imaginary Verma module**.

• Let $\lambda(c) = 0$. Denote by $J^q(\lambda)$ the left ideal of $U_q(\hat{\mathfrak{g}})$ generated by $x_{ik}^+, h_{il}$ for $i \in I_0, k, l \in \mathbb{Z}, l \neq 0$ and $K_i^{\pm 1} - q^{\pm\lambda(h_i)}$, $\gamma^{\pm 1/2} - 1$ and $D^{\pm 1} - q^{\pm\lambda(d)}$. Set

$$\tilde{M}_q(\lambda) = U_q(\hat{\mathfrak{g}})/J^q(\lambda)$$

It is a quotient of $M_q(\lambda)$, we call it **reduced imaginary Verma module**.

## Theorem [F., Grishkov, Melville]

Let $\lambda \in P$ such that $\lambda(c) = 0$. Then

- $\tilde{M}_q(\lambda)$ is irreducible if and only if $\lambda(h_i) \neq 0$ for all $i \in I_0$.
- $\tilde{M}_q(\lambda) = \bigoplus_{\substack{i_1,\dots,i_r \\ k_1,\dots,k_r}} \mathbb{C}(q^{1/2}) x_{i_1 k_1}^- \cdots x_{i_r k_r}^- v_\lambda$, where $v_\lambda$ stands for the generator of the module.

•$\mathrm{U}_q \cong U_q^-(S) \otimes U_q^0(S) \otimes U_q^+(S)$ and a PBW basis

## Kashiwara algebra

• Consider the subalgebra $\mathcal{N}_q^-$ of $U_q(\hat{\mathfrak{g}})$ generated by $\gamma^{\pm 1/2}$ and $x_{il}^-$ for $l \in \mathbb{Z}$, $i \in I_0$.

Set $\overline{P}^{j_1,\ldots,j_k} = x_{j_1}^-(v_1)\cdots x_{j_k}^-(v_k)$,
$\overline{P}_l^{j_1,\ldots,j_k} = x_{j_1}^-(v_1)\cdots x_{j_{l-1}}^-(v_{j_{l-1}})x_{j_{l+1}}^-(v_{j_l+1})\cdots x_{j_k}^-(v_k)$
and denote

$$G_{il} := \delta_{i,j_l}\prod_{m=1}^{l-1} g_{i,j_m,q^{-1}}(v_{j_m}/v_l), \quad G_{i1} = \delta_{i,j_1}$$

• We define operators $\Omega_i(k) : \mathcal{N}_q^- \longrightarrow \mathcal{N}_q^-$ for $k \in \mathbb{Z}$ in terms of the generating functions $\Omega_i(u) = \sum_{l\in\mathbb{Z}} \Omega_i(l)u^{-l}$:

$$\Omega_i(u)(\overline{P}^{j_1,\dots,j_k}) = \sum_{l=1}^{k} G_{il}\overline{P}_l^{j_1,\dots,j_k}\delta(u/v_l\gamma),$$

$\Omega_i(u)(1) = 0$. Also consider left multiplication operators $x_{im}^- : \mathcal{N}_q^- \longrightarrow \mathcal{N}_q^-$. The $\Omega$-operators and the $x^-$-operators satisfy the identities:

$$q^{(\alpha_i|\alpha_j)}\gamma\Omega_j(m)x_{i,n+1}^- - \Omega_j(m+1)x_{in}^- =$$
$$(q^{(\alpha_i|\alpha_j)}\gamma - 1)\delta_{ij}\delta_{m,-n-1} + \gamma x_{i,n+1}^-\Omega_j(m) - q^{(\alpha_1|\alpha_j)}x_{in}^-\Omega_j(m+1).$$

and

$$\Omega_j(k)x_{im}^- = \delta_{ij}\delta_{k,-m}\gamma^k + \sum_{r\geq 0} g_{i,j,q^{-1}}(r)x_{i,m+r}^-\Omega_j(k-r)\gamma^r.$$

• Define the Kashiwara algebra $\mathcal{K}_q$: the $\mathbb{C}(q^{1/2})$-algebra with generators $\Omega_j(m), x_i^-(n), \gamma^{\pm 1/2}$ for $m, n \in \mathbb{Z}$, $1 \leq i, j \leq N$, where $\gamma^{\pm 1/2}$ are central, $\gamma^{\pm 1/2}\gamma^{\mp 1/2} = 1$ and

$$q^{(\alpha_i|\alpha_j)}\gamma\Omega_j(m)x^-_{i,n+1} - \Omega_j(m+1)x^-_{in} =$$
$$(q^{(\alpha_i|\alpha_j)}\gamma - 1)\delta_{ij}\delta_{m,-n-1} + \gamma x^-_{i,n+1}\Omega_j(m) - q^{(\alpha_1|\alpha_j)}x^-_{in}\Omega_j(m+1)$$

$$q^{(\alpha_i|\alpha_j)}\Omega_i(k+1)\Omega_j(l) - \Omega_j(l)\Omega_i(k+1) =$$
$$\Omega_i(k)\Omega_j(l+1) - q^{(\alpha_i|\alpha_j)}\Omega_j(l+1)\Omega_i(k)$$

$$x^-_{i,k+1}x^-_{jl} - q^{-(\alpha_i|\alpha_j)}x^-_{jl}x^-_{i,k+1} = q^{-(\alpha_i|\alpha_j)}x^-_{ik}x^-_{j,l+1} - x^-_{j,l+1}x^-_{ik}$$

### Proposition

For a quantum affine algebra associated to any untwisted affine Lie algebra, there exists a unique non-degenerate symmetric form $(-,-)$ defined on $\mathcal{N}_q^-$ satisfying $(x_{ij}^- a, b) = (a, \Omega_i(-j)b)$ and $(1,1) = 1$. Moreover, $\mathcal{N}_q^-$ is a simple left $\mathcal{K}_q$-module such that

$$\mathcal{N}_q^- \cong \mathcal{K}_q / \Big( \sum_{i=1}^{N} \sum_{k \in \mathbb{Z}} \mathcal{K}_q \Omega_i(k) \Big)$$

**Remark:** 1) For Lie algebras of type ADE the result was shown by Cox, F., Misra, 2015.

2) This allows to define the Kashiwara-like operators on imaginary Verma modules.

• We say that a monomial $x^-_{i_1k_1}\cdots x^-_{i_rk_r}$ is ordered if $i_1+k_1\geq i_2+k_2\geq\cdots\geq i_r+k_r$

There exists a product, denoted $\star$, such that $x^-_{im}\star x^-_{jn}$ is ordered (for ordered monomials: $x^-_{i_1k_1}\cdots x^-_{i_lk_l}=x^-_{i_1k_1}\star\cdots\star x^-_{i_lk_l}$)

We define the operator $\tilde{x}^-_{jm}$ on $\star-$monomials as a $\star-$left multiplication. We also define the operator $\tilde{\Omega}_i(m)$ on $\star-$monomials inductively:

$$\tilde{\Omega}_i(m)(x^-_{jk}):=\delta_{ij}\delta_{-m,k},$$

$$\tilde{\Omega}_i(m)((x^-_{i_1k_1}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l})\cdots)))=$$

$$\delta_{ii_1}\delta_{-m,k_1}(x^-_{i_2k_2}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l})\cdots))$$

$$+\sum_{r\geq 0}q^{p^{mk_1}_{ii_1}}g_{i,i_1,q^{-1}}(r)(x^-_{i_1,m_1+r}\star\tilde{\Omega}_i(m-r)(x^-_{i_2k_2}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l})))$$

- Define the action of $\tilde{x}^-_{jm}$ and $\tilde{\Omega}_i(m)$ on $\tilde{M}_q(\lambda)$:

$$\tilde{x}^-_{jm}(x^-_{i_1k_1}\cdots x^-_{i_rk_r}v_\lambda)=\tilde{x}^-_{jm}(x^-_{i_1k_1}\star\cdots\star x^-_{i_rk_r}v_\lambda):=(x^-_{jm}\star x^-_{i_1k_1})\star\cdots\star x^-_{i_rk_r}v_\lambda$$

$$\begin{aligned}&\tilde{\Omega}_i(m)(x^-_{i_1k_1}\star\cdots\star x^-_{i_lk_l}v_\lambda)\\ &=\delta_{ii_1}\delta_{-m,k_1}x^-_{i_2k_2}\star\cdots\star x^-_{i_lk_l}v_\lambda\\ +\sum_{r\geq 0}q^{p^{mk_1}_{ii_1}}&g_{i,i_1,q^{-1}}(r)x_{i_1,m_1+r}\star\tilde{\Omega}_i(m-r)(x^-_{i_2k_2}\star\cdots\star x^-_{i_lk_l})v_\lambda\end{aligned}$$

- Set $\langle x^-_{im},x^-_{jn}\rangle:=(1,\tilde{\Omega}_i(-m)(x^-_{jn}))\in\mathbb{Z}[q]$

## Crystal-like basis

Let $\mathbb{A}_0 = \mathbb{C}[q^{1/2}]_{(q)}$ the ring of rational functions in $q^{1/2}$ regular at 0. Let $\Sigma = \{\mu - k\alpha + n\delta | \mu \in P, \alpha \in \Delta^+(\mathfrak{g}), k > 0, n \in \mathbb{Z}\} \cup \{\mu\}$.

• Let $M$ be a $U_q(\hat{\mathfrak{g}})$-module. We call a free $\mathbb{A}_0$-submodule $\mathcal{L}$ of $M$ an *imaginary crystal lattice* of $M$ if the following holds:

1. $\mathbb{C}(q^{1/2}) \otimes_{\mathbb{A}_0} \mathcal{L} \cong M$.
2. $\mathcal{L} \cong \oplus_{\lambda \in \Sigma} \mathcal{L}_\lambda$ and $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$.
3. $\tilde{\Omega}_{\psi_i}(m)\mathcal{L} \subseteq \mathcal{L}$ and $\tilde{x}^-_{im}\mathcal{L} \subseteq \mathcal{L}$, for $i \in I$ and $m \in \mathbb{Z}$.

## Crystal-like basis

Let $\mathbb{A}_0 = \mathbb{C}[q^{1/2}]_{(q)}$ the ring of rational functions in $q^{1/2}$ regular at 0. Let $\Sigma = \{\mu - k\alpha + n\delta | \mu \in P, \alpha \in \Delta^+(\mathfrak{g}), k > 0, n \in \mathbb{Z}\} \cup \{\mu\}$.

- Let $M$ be a $U_q(\hat{\mathfrak{g}})$-module. We call a free $\mathbb{A}_0$-submodule $\mathcal{L}$ of $M$ an *imaginary crystal lattice* of $M$ if the following holds:
  1. $\mathbb{C}(q^{1/2}) \otimes_{\mathbb{A}_0} \mathcal{L} \cong M$.
  2. $\mathcal{L} \cong \oplus_{\lambda \in \Sigma} \mathcal{L}_\lambda$ and $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$.
  3. $\tilde{\Omega}_{\psi_i}(m)\mathcal{L} \subseteq \mathcal{L}$ and $\tilde{x}_{im}^- \mathcal{L} \subseteq \mathcal{L}$, for $i \in I$ and $m \in \mathbb{Z}$.
- The $\mathbb{A}_0$-module

$$\mathcal{L}(\lambda) := \bigoplus_{\substack{k \geq 0 \\ i_1 + m_1 \geq \cdots \geq i_k + m_k \\ i_l, m_l \in \mathbb{Z}}} \mathbb{A}_0 x_{i_1 m_1}^- \cdots x_{i_k m_k}^- v_\lambda$$

is an imaginary crystal lattice of the irreducible reduced imaginary Verma module $\tilde{M}_q(\lambda)$. Moreover,
$\mathcal{L}(\lambda) = \{u \in \tilde{M}_q(\lambda) | \langle u, \mathcal{L}(\lambda) \rangle \subset \mathbb{A}_0\}$

• An *imaginary crystal base* of an irreducible reduced imaginary Verma module $\tilde{M}_q(\lambda)$ is a pair $(\mathcal{L}, \mathcal{B})$ satisfying:

1. $\mathcal{L}$ is an imaginary crystal lattice of $M$.
2. $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}_0} \mathcal{L}$.
3. $\mathcal{B} = \cup_{\mu \in \Sigma} \mathcal{B}_\mu$ where $\mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu / q\mathcal{L}_\mu)$.
4. $\tilde{x}_{im}^{-}\mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$ and $\tilde{\Omega}_i(m)\mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$.
5. For $m \in \mathbb{Z}$ and $i \in I_0$ if $\tilde{\Omega}_i(-m)b \neq 0$ and $\tilde{x}_{im}^{-}b \neq 0$ for $b \in \mathcal{B}$, then $\tilde{x}_{im}^{-}\tilde{\Omega}_i(-m)b = \tilde{\Omega}_i(-m)\tilde{x}_{im}^{-}b$.

- An *imaginary crystal base* of an irreducible reduced imaginary Verma module $\tilde{M}_q(\lambda)$ is a pair $(\mathcal{L}, \mathcal{B})$ satisfying:
  1. $\mathcal{L}$ is an imaginary crystal lattice of $M$.
  2. $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}_0} \mathcal{L}$.
  3. $\mathcal{B} = \cup_{\mu \in \Sigma} \mathcal{B}_\mu$ where $\mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu / q\mathcal{L}_\mu)$.
  4. $\tilde{x}_{im}^- \mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$ and $\tilde{\Omega}_i(m)\mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$.
  5. For $m \in \mathbb{Z}$ and $i \in I_0$ if $\tilde{\Omega}_i(-m)b \neq 0$ and $\tilde{x}_{im}^- b \neq 0$ for $b \in \mathcal{B}$, then $\tilde{x}_{im}^- \tilde{\Omega}_i(-m)b = \tilde{\Omega}_i(-m)\tilde{x}_{im}^- b$.
- For $\lambda \in \hat{\mathfrak{h}}^*$ define

$$\mathcal{B}(\lambda) = \left\{ x_{i_1 m_1}^- \cdots x_{i_k m_k}^- v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \right.$$

$$\left| \begin{array}{c} i_1 + m_1 \geq \cdots \geq i_k + m_k, \\ m_1, \ldots, m_k \in \mathbb{Z}, i_1, \ldots, i_k \in I_0 \end{array} \right\}$$

## Theorem [Arias, V.F., Misra, 2024]

If $\lambda \in \hat{\mathfrak{h}}^*$ such that $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$ for all $i \in I_0$, then $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is an imaginary crystal base for $\tilde{M}_q(\lambda)$.

## Theorem [Arias, V.F., Misra, 2024]

If $\lambda \in \hat{\mathfrak{h}}^*$ such that $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$ for all $i \in I_0$, then $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is an imaginary crystal base for $\tilde{M}_q(\lambda)$.

- Let $\hat{\mathfrak{h}}^*_{red} = \{\lambda \in P \mid \lambda(c) = 0, \lambda(h_i) \neq 0, i \in I_0\}$. Consider the category $\mathcal{O}^q_{red,im}$ of $U_q(\hat{\mathfrak{g}})$-modules $M$ such that

1. $M$ is a weight module with weights in $\hat{\mathfrak{h}}^*_{red}$.
2. For any $i \in I_0$ and any $n \in \mathbb{Z}$, $x^+_{in}$ is locally nilpotent.
3. $M$ is $G_q$-compatible.

We have in $\mathcal{O}^q_{red,im}$ (Arias, V.F., Oliveira, 2025):

(1) If $\lambda, \mu \in \hat{\mathfrak{h}}^*_{red}$ then $\mathrm{Ext}^1_{\mathcal{O}^q_{red,im}}(\tilde{M}_q(\lambda), \tilde{M}_q(\mu)) = 0$.

(2) If $M$ is an irreducible module in the category $\mathcal{O}^q_{red,im}$, then $M \cong \tilde{M}_q(\lambda)$ for some $\lambda \in \hat{\mathfrak{h}}^*_{red}$. Moreover, if $N$ is an arbitrary object of $\mathcal{O}^q_{red,im}$ then $N \cong \bigoplus_{\lambda_i \in \hat{\mathfrak{h}}^*_{red}} \tilde{M}(\lambda_i)$, for some $\lambda_i' s$.

Theorem [Arias, V.F., Misra, 2024]

The operators $\tilde{x}_{in}^{-}$ and $\tilde{\Omega}_i(m)$ are well defined on objects of the category $\mathcal{O}^q_{red,im}$. Every object of the category has an imaginary crystal basis.

**Remark:** For $\mathfrak{g} = sl(2)$ the result was shown by Cox, F., Misra, 2017.

Thank you!