On $U(\mathfrak{h})$-Free $\mathfrak{sl}(2)$-Modules of Finite Rank

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Conventions and basic definitions

- The ground field is $\mathbb{C}$.
- $\mathfrak{s}$ will stand for a simple complex finite-dimensional Lie algebra. For example, $\mathfrak{s} = \mathfrak{sl}(n)$ and $\mathfrak{s} = \mathfrak{sp}(2n)$.
- We will focus mostly on $\mathfrak{s} = \mathfrak{sl}(2)$.
- $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{s}$. For $\mathfrak{s} = \mathfrak{sl}(2)$, $\mathfrak{s}$ consists of the diagonal matrices in $\mathfrak{sl}(2)$.
- By $U(\mathfrak{a})$ we denote the universal envelopping algebra of the Lie algebra $\mathfrak{a}$.

Weight and $\mathfrak{h}$-free modules

- An $\mathfrak{s}$-module M is a *weight module of finite type* if

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda, \text{ and } \dim M^\lambda < \infty$$

where $M^\lambda := \{m \in M \mid h \cdot m = \lambda(h)m, \text{ for every } h \in \mathfrak{h}\}$.

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- Simple weight $\mathfrak{s}$-modules of finite type are classified by O. Mathieu in 2000 based on earlier works of Benkart, Britten, Fernando, Futorny, Lemire, others.
- The case $\mathfrak{s} = \mathfrak{sl}(2)$ is known since the 1960s (Drozd, Gabirel, Dixmier). Excellent reference: "Lectures on $\mathfrak{sl}_2(\mathbb{C})$-modules" by V. Mazorchuk.

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- An $\mathfrak{s}$-module M is *$U(\mathfrak{h})$-free of rank n* if $\mathrm{Res}_{U(\mathfrak{h})}^{U(\mathfrak{s})} M \simeq U(\mathfrak{h})^{\oplus n}$, i.e., if $M \simeq \mathbb{C}[\mathfrak{h}]^{\oplus n}$ as a module over $\mathfrak{h}$.

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- Goal: study $U(\mathfrak{h})$-free $\mathfrak{sl}(2)$-modules of finite rank.

Weighting functor and examples

- For any module M over $\mathfrak{s}$, the *weighting* $\mathcal{W}(M)$ of M is a weight module defined as follows. If $\lambda \in \mathfrak{h}^*$ by $\overline{\lambda} : U(\mathfrak{h}) \to \mathbb{C}$ we denote the homomorphism such that $\overline{\lambda}|\mathfrak{h} = \lambda$. Then

$$\mathcal{W}(M) := \bigoplus_{\mathfrak{m} \in \mathrm{Max}\, U(\mathfrak{h})} M/\mathfrak{m}M = \bigoplus_{\lambda \in \mathfrak{h}^*} M/\ker(\overline{\lambda})M$$

  is $\mathfrak{s}$-module via $x_\alpha \cdot (v + \ker(\overline{\lambda})M) := (x_\alpha \cdot v) + \ker(\overline{\lambda + \alpha})M$, where x_α is in the α-root space of $\mathfrak{s}$.

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- Let $\mathfrak{s} = \mathfrak{sl}(2)$, $a \in \mathbb{C}$, and $g \in \mathbb{C}[t]$ with $\deg g = n$. By $E(g, a)$, we denote the space $\mathbb{C}[t]e^g$, equipped with the following $\mathfrak{sl}(2)$-action:

$$\begin{aligned} e &\mapsto t^2\partial_t + 2at, \\ f &\mapsto -\partial_t, \\ h &\mapsto t\partial_t + a\mathrm{Id}. \end{aligned}$$

  Then $E(g, a)$ is a $U(\mathfrak{h})$-free module of rank n. Moreover, if $g(t) = a_1 t + a_2 t^2$, $a_2 \neq 0$, then $E(g, a)$ is simple.

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  Then $E(g, a)$ is a $U(\mathfrak{h})$-free module of rank n. Moreover, if $g(t) = a_1 t + a_2 t^2$, $a_2 \neq 0$, then $E(g, a)$ is simple.
- $\mathcal{W}(E(g, a))$ is a coherent family of degree n.

Further conventions and definitions for $\mathfrak{s} = \mathfrak{sl}(2)$

From now on $\mathfrak{s} = \mathfrak{sl}(2)$!

- We fix basis $\{e, f, h\}$, where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

- As a result: $[h, e] = e, \quad [e, f] = 2h, \quad [h, f] = -f.$

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Definition (σ–similarity and σ^{-1}–similarity)

Let $A, B \in \mathrm{Mat}_n(\mathbb{C}[h])$. We say that A and B are *σ-similar*, denoted $A \overset{\sigma}{\sim} B$, if there exists $P(h) \in \mathrm{GL}_n(\mathbb{C}[h])$ such that

$$B = P^{-1}(h) A P(h-1).$$

Similarly, we define *σ^{-1}-similarity*.

The Modules $M(E, F)$

Lemma

Let M be $\mathfrak{sl}(2)$-module that is $\mathbb{C}[h]$-free of rank n. Then $M = M(E, F)$, where $M(E, F)$ is defined as follows.

- $M(E, F) = \mathbb{C}[h]^{\oplus n}$ *as a vector space.*

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- *$M(E, F) = \mathbb{C}[h]^{\oplus n}$ as a vector space.*
- *$E, F \in \mathrm{Mat}_n(\mathbb{C}[h])$ are such that*

$$E\sigma(F) - F\sigma^{-1}(E) = 2h\mathrm{Id}_n.$$

- *The $\mathfrak{sl}(2)$-module structure on $M(E, F)$ is given by:*

$$e\cdot \begin{pmatrix} g_1(h) \\ g_2(h) \\ \vdots \\ g_n(h) \end{pmatrix} = E \begin{pmatrix} \sigma(g_1(h)) \\ \sigma(g_2(h)) \\ \vdots \\ \sigma(g_n(h)) \end{pmatrix}, \quad f\cdot \begin{pmatrix} g_1(h) \\ g_2(h) \\ \vdots \\ g_n(h) \end{pmatrix} = F \begin{pmatrix} \sigma^{-1}(g_1(h)) \\ \sigma^{-1}(g_2(h)) \\ \vdots \\ \sigma^{-1}(g_n(h)) \end{pmatrix},$$

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Lemma

$M(E, F) \simeq M(E', F')$ if and only if there exists $P \in \mathrm{GL}_n(\mathbb{C}[h])$ such that: $E' = P^{-1}(h)\, E\, P(h-1), \quad F' = P^{-1}(h)\, F\, P(h+1).$

$U(\mathfrak{h})$-free modules with central character

Theorem

Let M be a $\mathcal{U}(\mathfrak{h})$-free module of rank n that admits a central character $\gamma = (2\alpha - 1)^2$. Then $M \cong M(E, F)$, where

$$F = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_n \end{pmatrix} K(h), \quad E = \sigma\left(K^{-1}(h) \begin{pmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_n \end{pmatrix} \right),$$

for some $K(h) \in \mathrm{GL}_n(\mathbb{C}[h])$ and $\mu_i, \xi_i \in \mathbb{C}[h]$ satisfying the following conditions:

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- $\mu_i \xi_i = -(h - \alpha + 1)(h + \alpha) =: \lambda_\alpha(h)$ *for all $i \in \{1, \ldots, n\}$.*

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Remark

We have:
$\mathrm{SNF}(F) = \mathrm{diag}(\mu) = P(h)FQ(h)$; $K(h) = Q^{-1}(h)P^{-1}(h+1)$.

Further Definitions an Notation

Definition

Let $\mathbf{a} = (a_-, a_0, a_+)$ be a triple with $a_-, a_+ \in \mathbb{Z}_{\geq 0}$ and $a_0 \in \mathbb{Z}$ such that $a_- + |a_0| + a_+ = n$. We define the diagonal matrix $P_{(\mathbf{a},\alpha)}(h) \in \mathrm{Mat}_n(\mathbb{C}[h])$ as follows:

(i) $P_{(\mathbf{a},\alpha)}(x)_{ii} = 1$ for $i = 1, \ldots, a_-$,

(ii) If $a_0 \geq 0$, then $P_{(\mathbf{a},\alpha)}(h)_{ii} = h - \alpha + 1$ for $i = a_- + 1, \ldots, a_- + a_0$,

(iii) If $a_0 < 0$, then $P_{(\mathbf{a},\alpha)}(h)_{ii} = h + \alpha$ for $i = a_- + 1, \ldots, a_- - a_0$,

(iv) $P_{(\mathbf{a},\alpha)}(h)_{ii} = (h - \alpha + 1)(h + \alpha)$ for $i = a_- + |a_0| + 1, \ldots, n$.

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Definition

If $E = P_{(\mathbf{a},\alpha)}(h)$, then we denote $M(E, F)$ by

$$M\left(\alpha, \mathbf{a}, K(h)\right),$$

or simply by $M\left(\mathbf{a}, K\right)$ if α (hence, the central character) is fixed.
Note: $M\left(\alpha, \mathbf{a}, K(h)\right) \simeq M\left(1-\alpha, \mathbf{a}', K(h)\right)$ for $\mathbf{a}' = (a_-, -a_0, a_+)$.

Exponential Modules as $M(\mathbf{a}, K)$

Proposition (G., Nguyen, Zhao, 2026)

Let $a_n \in \mathbb{C}^*$, and $a_i \in \mathbb{C}$, for all $i \in \{1, \cdots, n-1\}$. Then the exponential module $E\left(\sum_{i=1}^{n} a_i t^i, \alpha\right)$ is isomorphic to

$$M\left(P_{((1,n-1,0),\alpha)}(h), \begin{pmatrix} a_1 & -1 & 0 & \cdots & 0 \\ 2a_2 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (n-1)a_{n-1} & 0 & \cdots & 0 & -1 \\ na_n & 0 & \cdots & 0 & 0 \end{pmatrix}\right).$$

The Rank-One Case

Theorem (Nilsson, 2015)

Let $\alpha \in \mathbb{C}$ *and let* $\beta \in \mathbb{C}^*$. *Then*

1. *The modules* $M(\alpha, (1,0,0), \beta)$ *and* $M(\alpha, (0,0,1), \beta)$ *are simple.*

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1. *The modules $M(\alpha, (1, 0, 0), \beta)$ and $M(\alpha, (0, 0, 1), \beta)$ are simple.*
2. *The module $M(\alpha, (0, 1, 0), \beta)$ is simple if and only if $\alpha \notin \frac{1}{2}\mathbb{Z}_{\geq 0} + 1$. When $\alpha \in \frac{1}{2}\mathbb{Z}_{\geq 0} + 1$, we have the following non-split short exact sequence:*

$$0 \longrightarrow M(\alpha, (0, -1, 0), \beta) \longrightarrow M(\alpha, (0, 1, 0), \beta) \longrightarrow L(2\alpha - 1) \longrightarrow 0.$$

Simple Rank-Two Free $U(\mathfrak{h})$-Modules of Scalar Type

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For $u(h) \in \mathbb{C}[h]$, set

$$E(u(h)) := \begin{pmatrix} u(h) & 1 \\ -1 & 0 \end{pmatrix}.$$

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Theorem (G., Ngueyn, Zhao, 2026)

Let $\mathbf{a} \in \{(2,0,0),(0,2,0),(0,0,2)\}$, *i.e.,* $\mu_1 = \mu_2$ *and* $\xi_1 = \xi_2$. *The module* $M(\alpha, \mathbf{a}, K(h))$ *is simple if and only if*

$$K(h) \overset{\sigma^{-1}}{\sim} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} E(u_1(h)) \cdots E(u_k(h)),$$

where $k \geq 1$, $a, b \in \mathbb{C}^*$, $u_i(h) \in \mathbb{C}[h] \setminus \mathbb{C}$, *and* α *and* $\mathbf{a}$ *are:*

1. *Any* $\alpha \in \mathbb{C}$, *if* $\mathbf{a} = (2,0,0)$ *or* $\mathbf{a} = (0,0,2)$,
2. $\alpha \in \mathbb{C} \setminus \left(\frac{1}{2}\mathbb{Z}_{\geq 0} + 1\right)$, *if* $\mathbf{a} = (0,2,0)$.

Remarks

(i) The description of all $K(h) \in \mathrm{GL}_2(\mathbb{C}[h])$ that are σ^{-1}-similar to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ (exactly when $M(\mathbf{a}, K)$ has a rank-1 submodule) is a nontrivial question and uses the standard form of K in $\mathrm{GL}_2(\mathbb{C}[h])$ (P.M. Cohn, 1966).

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(ii) The rank-two case when **a** is not of scalar type remains open. In other words, we still do not know what rank-two modules $M(E, F)$ are simple when $\deg \mu_1 < \deg \mu_2$.

Remarks

(i) The description of all $K(h) \in \mathrm{GL}_2(\mathbb{C}[h])$ that are σ^{-1}-similar to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ (exactly when $M(\mathbf{a}, K)$ has a rank-1 submodule) is a nontrivial question and uses the standard form of K in $\mathrm{GL}_2(\mathbb{C}[h])$ (P.M. Cohn, 1966).

(ii) The rank-two case when $\mathbf{a}$ is not of scalar type remains open. In other words, we still do not know what rank-two modules $M(E, F)$ are simple when $\deg \mu_1 < \deg \mu_2$.

(iii) Examples of $U(\mathfrak{h})$-free $\mathfrak{sl}(2)$-modules that have been previously studied (all of them are modules $M(\mathbf{a}, K)$):

   (a) Rank 1 modules of J. Nilsson (2015, 2016).

   (b) Rank n modules of F. Martin, C. Prieto (2017) $\mathbf{a} = (n, 0, 0)$

   (c) Rank 2 Modules of Y. Bahturin , A. Shihadeh (2022). $\mathbf{a} = (1, 0, 1)$

   (d) Exponential tensor modules (G., Nguyen, 2022).

   (e) Exponential modules over Smith algebras (Futorny, Lopes, Mendonça, 2023). $\mathbf{a} = (0, n-1, 1), (1, n-1, 0)$

   (f) Free modules over GWAs (Lopes, Nilsson, 2025). $\mathbf{a} = (n-1, 1, 0)$

Simplified version of the classification foscalar type

Theorem (Cohn)

Every $A \in \mathrm{GL}_2(\mathbb{C}[h])$ *admits a unique factorization of the form*

$$A = \mathrm{diag}(\beta_1, \beta_2) \qquad \text{or} \qquad A = \mathrm{diag}(\beta_1, \beta_2) \prod_{i=1}^{k} E\big(u_i(h)\big),$$

for some $k \geq 1$, $\beta_1, \beta_2 \in \mathbb{C}^*$, $u_i(h) \in \mathbb{C}[h]$, *and* $u_j \notin \mathbb{C}^*$, $1 < j < k$.

Lemma

Let $a, b \in \mathbb{C}^*$. *Every* σ^{-1}*-similar matrix to* $\mathrm{diag}(a, b)$ *has the form*

$$(-1)^k\, \mathrm{diag}(a, b)_{[k]}\, E(0)\, E\left(-\left(\tfrac{a}{b}\right)^{(-1)^k} u_k\right) \cdots E\left(-\tfrac{a}{b}\, u_2\right) E\left(-\tfrac{b}{a}\, u_1 + u_1(h+1)\right)\, E\big(u_2(h+1)\big) \cdots E\big(u_k(h+1)\big),$$

for some $k \in \mathbb{Z}_{\geq 1}$ *and* $u_i \in \mathbb{C}[h]$ *for* $1 \leq i \leq k$, *with* $u_j(h) \notin \mathbb{C}$ *for* $1 < j < k$.

Remark. The presentation of $K(h)$ in Lemma is not necessarily in standard form - take for instance $u_1(h) = 0$ with $k \geq 1$.

Weighting of $M(\alpha, \mathbf{a}, K)$

We introduce 4 coherent families of $\mathfrak{sl}(2)$, all with underlying space $Span\{v_\lambda \mid \lambda \in \mathbb{C}\}$ on which the action of e, f, h is as follows.

(i) The family $\mathcal{F}_-(\alpha)$:

$$\begin{aligned} e(v_\lambda) &= (\alpha - \lambda)(\alpha + 1 - \lambda)v_{\lambda+1} \\ fv_\lambda &= v_{\lambda-1} \\ hv_\lambda &= \lambda v_\lambda \end{aligned}$$

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(iii) $\mathcal{F}_0^+(\alpha) = \mathcal{F}_0^-(\alpha)^\vee$.

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Theorem (G., Nguyen, Zhao, 2026)

$$\mathcal{W}(M((a_-, a_0, a_+), K)) \simeq a_-\mathcal{F}_- \oplus a_0\mathcal{F}_0 \oplus a_+\mathcal{F}_+,$$

where $a_0\mathcal{F}_0 = -a_0\mathcal{F}_0^-$ *if* $a_0 < 0$, *and* $a_0\mathcal{F}_0 = a_0\mathcal{F}_0^+$ *if* $a_0 \geq 0$.

Socle Filtration of $M(\mathbf{a}, J_n)$

We consider the case when K equals a single Jordan cell J_n.

Theorem (G., Nguyen, Zhao, 2026)

Let $n > 1$. Then, the layers of the socle filtration of $M((a_-, a_0, a_+), J_n)$ are as follows

$$\begin{aligned} \mathrm{soc}_1 M((a_-, a_0, a_+), J_n) &= M((1,0,0),1)^{\oplus a_-}, \\ \mathrm{soc}_2 M((a_-, a_0, a_+), J_n) &= M((0,1,0),1)^{\oplus |a_0|} \\ \mathrm{soc}_3 M((a_-, a_0, a_+), J_n) &= M((0,0,1),1)^{\oplus a_+}. \end{aligned}$$

THANK YOU

Lemma (Cohn)

For $u(h), v(h), w(h) \in \mathbb{C}[h]$ and $\beta_1, \beta_2 \in \mathbb{C}^*$, the following identities hold:

(i) $E(u(h))E(0)E(v(h)) = -E(u(h)+v(h))$,

(ii) $E(\beta_1)E\left(\beta_1^{-1}\right)E(\beta_1) = -\mathrm{diag}\left(\beta_1, \beta_1^{-1}\right)$,

(iii) $E(u(h))\mathrm{diag}(\beta_1, \beta_2) = \mathrm{diag}(\beta_2, \beta_1)E\left(\frac{\beta_1}{\beta_2}u(h)\right)$,

(iv) $E(u(h))E(v(h))^{-1} = E(u(h)-v(h))E(0)^{-1} = -E(u(h)-v(h))E(0)$,

(v) $E(u(h))E(v(h))^{-1}E(w(h)) = E(u(h)-v(h)+w(h))$,

(vi) $E(u(h))E(\beta_1)E(v(h)) = E\left(u(h)-\beta_1^{-1}\right)\mathrm{diag}\left(\beta_1, \beta_1^{-1}\right)E\left(v(h)-\beta_1^{-1}\right)$.