$*$-algebra structures on path algebras

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Motivation from noncommutative geometry

In geometry, one can represent a geometric space by its (commutative) algebra of functions. A lot of geometric concepts can then be restated in an algebraic way, for example vector bundles are exactly the projective modules over the algebra of functions. The idea of noncommutative geometry is to study “noncommutative spaces” by studying the geometry of noncommutative algebras.

From this simple idea many different approaches to noncommutative geometry has been developed. In one such approach, introduced by Dubois–Violette (1988), one starts with a $*$-algebra A (representing our space) and a Lie algebra $\mathfrak{g} \subset \mathrm{Der}(A)$ (representing the tangent bundle), and constructs a dg-algebra $\Omega_{\mathfrak{g}}$ (representing the cotangent bundle) with the goal to study noncommutative complex geometry.

Motivation from noncommutative geometry

Previously, Arnlind (2024) has studied the geometry of the path algebra of the (genalized) Kronecker quiver. Motivated by the work of Arnlind, we want to study the noncommutative geometry of quiver path algebras in greater generality.

Then the question of existence of a $*$-algebra structure arises. Is there some structure on the quiver that is in correspondence with $*$-algebra structures on the path algebra? Can the existence and other properties of such a structure easily be determined by the shape of the quiver?

Quivers and path algebras

A *quiver* is a directed graph $Q = (Q_0, Q_1, t, h)$ where Q_0 is the set of *vertices*, Q_1 is the set of *arrows*, and for an arrow $\alpha \in Q_1$ the vertice $t(\alpha)$ is the *tail* and $h(\alpha)$ is the *head*, i.e. $\alpha : t(\alpha) \to h(\alpha)$. Its *opposite quiver* Q^{op} switches the places of $t(\alpha)$ and $h(\alpha)$. We assume the sets Q_0, Q_1 to be finite.

A *path* p in Q is a sequence of arrows $p = \alpha_n \dots \alpha_1$ such that $t(\alpha_i) = h(\alpha_{i-1})$. We also include a path of length zero e_i for each vertex $i \in Q_0$. This zero path is an idempotent.

The *path algebra* $\mathbb{C}Q$ of Q is the vector space

$$\mathbb{C}Q = \bigoplus_{p \text{ path in } Q} \mathbb{C}p$$

with multiplication defined by concatenation of paths where possible and zero elsewhere. The path algebra is an associative algebra and it is finite dimensional if and only if the quiver contains no cycles.

Example

Consider the quiver $Q : 1 \xrightarrow{\alpha} 2$. The path algebra of Q is generated by $\{e_1, e_2, \alpha\}$ with multiplication table

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_1	α
α	α	0	0

The path algebra $\mathbb{C}Q$ is isomorphic to the algebra of lower triangular 2×2 matrices $L_2(\mathbb{C}) = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$ via

$$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$*$-algebras

A *$*$-algebra* $(A, *)$ is a associative unital $\mathbb{C}$-algebra A together with a map $* : A \to A$ satisfying

$$(a^*)^* = a, \quad (\lambda a + b)^* = \overline{\lambda} a^* + b^*, \quad (ab)^* = b^* a^*, \quad 1_A^* = 1_A$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

Example

- The full matrix algebra $M_n(\mathbb{C})$ with the conjugate transpose is a $*$-algebra.
- The algebra of lower triangular 2×2 matrices $L_2(\mathbb{C})$ with $*$-structure defined on generators by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Quasi–symmetric quivers

Note that for a $*$-algebra A the $*$-structure is an (anti-linear) isomorphism of algebras $A \to A^{op}$, and that $(\mathbb{C}Q)^{op} \simeq \mathbb{C}(Q^{op})$. We want a corresponding structure on quivers.

A *quasi–symmetric quiver* is a quiver Q together with an involution $\sigma : Q_0 \sqcup Q_1 \to Q_0 \sqcup Q_1$ satisfying

(i) $\sigma(Q_0) = Q_0$ and $\sigma(Q_1) = Q_1$; and

(ii) $\sigma(h\alpha) = t\sigma(\alpha)$ and $\sigma(t\alpha) = h\sigma(\alpha)$ for all $\alpha \in Q_1$.

Example

Consider the quiver $Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, and let $\sigma(1) = 2$ and $\sigma(\alpha) = \beta$. Then (Q, σ) is a quasi–symmetric quiver, e.g.

$$\sigma(t\alpha) = \sigma(1) = 2 = h\beta = h\sigma(\alpha).$$

$*$-path algebras

A quasi–symmetry on a quiver Q is exactly an involutive isomorphism of quivers $Q \to Q^{op}$.

Proposition

If (Q, σ) is a quasi–symmetric quiver then $\mathbb{C}Q$ can be equipped with a $*$-structure. Conversely, if $\mathbb{C}Q$ is a $*$-algebra such that $*(Q_0) = Q_0$ and $*(Q_1) = Q_1$ then Q can be equipped with a quasi–symmetry.

Example

Consider the quasi–symmetric quiver

$$Q : 1 \xrightarrow{\alpha} 2, \quad \sigma(1) = 2 \quad \text{and} \quad \sigma(\alpha) = \alpha.$$

The $*$-structure on $\mathbb{C}Q$ is then given on generators by $e_1^* = e_{\sigma(1)} = e_2$ and $\alpha^* = \sigma(\alpha) = \alpha$.

$*$-path algebras

Not all $*$-structures on path algebras are induced by quasi–symmetries.

Non–Example

Consider the quiver $Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, and define $*$ by

$$\alpha^* = \beta, \quad e_1^* = e_2 - \alpha - \beta, \quad \text{and} \quad e_2^* = e_1 + \alpha + \beta.$$

Then $\mathbb{C}Q$ is a $*$-algebra, e.g.

$$\begin{aligned}(e_1^*)^* = e_2^* - \alpha^* - \beta^* = e_1 + \alpha + \beta - \beta - \alpha = e_1,\\ e_1^*\alpha^* = (e_2 - \alpha - \beta)\beta = \beta = \alpha^* = (\alpha e_1)^*,\\ e_2^*\beta^* = (e_1 + \alpha + \beta)\alpha = 0 = (\beta e_2)^*,\end{aligned}$$

but $*(Q_0) \neq Q_0$. Note, however, that Q can still be equipped with a quasi–symmetry.

Inducing a quasi–symmetry

Note that any $*$-structure induces an equivalence functor: let A be an $*$-algebra and M a left A-module. The left A^{op}-module $\widehat{M}$ has same underlying abelian group as M, and its A^{op}-structure is defined by

$$a \cdot \widehat{m} = \widehat{a^* m}, \quad a \in A^{op}, \quad \widehat{m} \in \widehat{M}.$$

Lemma (Beggs–Majid, 2009)

Let $H_A : \text{Mod } A \to \text{Mod } A^{op}$ be defined by

$$M \mapsto \widehat{M} \quad \textit{and} \quad f \mapsto f.$$

Then H_A is a conjugate linear isomorphism of categories, with $H_{A^{op}}$ as its inverse.

Inducing a quasi–symmetry

Let Q be an acyclic quiver and $A = \mathbb{C}Q$ its path algebra. Then

- the set of vertices in Q are in bijection with (isomorphism classes of) simple A-modules $\{S_1, \ldots, S_n\}$;
- the number of arrows $i \to j$ is $\dim_{\mathbb{C}} \mathrm{Ext}^1_A(S_i, S_j)$.

Assuming that A is a $*$-algebra, then the equivalence functor $H_A : \mathrm{Mod}\, A \to \mathrm{Mod}\, A^{op}$ induces a bijection between isomorphism classes of simple A-modules $[S_i]$ and isomorphism classes of simple A^{op}-modules $[\Sigma_{k_i}]$, as well as a $\mathbb{C}$-linear isomorphism

$$\mathrm{Ext}^1_A(S_i, S_j) \simeq \mathrm{Ext}^1_{A^{op}}(\Sigma_{k_i}, \Sigma_{k_j}).$$

Thus the number of arrows $i \to j$ in Q is the same as the number of arrows $k_i \to k_j$ in Q^{op} (i.e. the number of arrows $k_j \to k_i$ in Q).

Inducing a quasi–symmetry

One can then for every $\alpha : i \to j$ choose a unique $\beta_\alpha : k_j \to k_i$. Defining $\sigma : Q \to Q$ by

$$\sigma(i) = k_i \quad \text{and} \quad \sigma(\alpha) = \beta_\alpha,$$

we get a quasi–symmetric quiver (Q, σ).

Combining this with the previous proposition, we get the following.

Theorem

Let Q be an (acyclic) quiver and let $A = \mathbb{C}Q$. Then A admits a $*$-algebra structure if and only if Q admits a quasi–symmetric structure.

Remark

This theorem holds even for non–acyclic quivers, but then one has to be careful since then the set of vertices are not in bijection with (isomorphism classes of) simple modules.

References

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Tack!