Recognition Theorems for Triangulated Categories

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Some Conventions

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All categories will be assumed to be $\Bbbk$-linear with finite-dimensional Hom-spaces and split idempotents.

Let $\mathcal{T}$ be a triangulated category.

- We denote the suspension functor by $[1]$.
- When it exists, we let ν denote the Serre functor on $\mathcal{T}$, that is a triangle autoequivalence characterized by the existence of a natural isomorphism:
$$\mathrm{Hom}(X, Y) \simeq D\mathrm{Hom}(Y, \nu X)$$
where $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ denotes the $\Bbbk$-dual.

d-Cluster Tilting Subcategories

Definition

Let $\mathcal{T}$ be a triangulated category. A functorially finite subcategory $\mathcal{C} \subset \mathcal{T}$ is called a d-cluster tilting subcategory if it satisfies:

$$\begin{aligned} \mathcal{C} &= \operatorname{add}\{X \in \mathcal{T} : \operatorname{Hom}(X, \mathcal{C}[i]) = 0,\ 0 < i < d\} \\ &= \operatorname{add}\{X \in \mathcal{T} : \operatorname{Hom}(\mathcal{C}, X[i]) = 0,\ 0 < i < d\} \end{aligned}$$

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- If $\mathcal{T}$ admits a Serre functor, then any d-cluster tilting subcategory $\mathcal{C} \subset \mathcal{T}$ satisfies $\mathcal{C} = \nu_d\mathcal{C} = \nu_d^{-1}\mathcal{C}$ where $\nu_d := \nu \circ [-d]$.

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- Any d-cluster tilting subcategory $\mathcal{C} \subset \mathcal{T}$ generates $\mathcal{T}$ in the sense that:

$$\mathcal{T} = \mathcal{C} * \mathcal{C}[1] * \cdots * \mathcal{C}[d-1]$$

The Problem

Question: To what extent does a d-cluster tilting subcategory determine its ambient triangulated category?

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More precisely, if $\mathcal{C} \subset \mathcal{T}$ and $\mathcal{C}' \subset \mathcal{T}'$ are d-cluster tilting subcategories and $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence, can we find a triangle equivalence $\widehat{F} : \mathcal{T} \to \mathcal{T}'$ such that the following diagram commutes?

$$\begin{tikzcd} \mathcal{T} \arrow[r, dashed, "\widehat{F}"] & \mathcal{T}' \\ \mathcal{C} \arrow[u, hook] \arrow[r, "F"'] & \mathcal{C}' \arrow[u, hook] \end{tikzcd}$$

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Answer: In general, this is not possible. The notion of d-cluster tilting subcategory is too weak. We require something stronger.

The "Vosnex" Property

Definition.

A d-cluster tilting subcategory $\mathcal{C}$ is said to satisfy the **vanishing of small negative extensions property** if

$$\mathrm{Hom}(X, Y[-i]) = 0$$

for all $X, Y \in \mathcal{C}$ and $0 < i < d - 1$.

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Example (Keller-Reiten, 2008)

Let Λ be a finite-dimensional hereditary algebra. Then

$$\mathcal{U}_d(\Lambda) := \mathrm{add}\{\nu_d^n \Lambda | n \in \mathbb{Z}\}$$

is a vosnex d-cluster tilting subcategory of $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$.

Previous Results: Cluster Categories of Hereditary Algebras

Definition

For a finite dimensional hereditary algebra Λ, the d-cluster category $\mathcal{C}^d(\Lambda)$ is defined to be the orbit category of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ with respect to the functor ν_d.

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- Objects are the same as $\mathcal{D}^b(\operatorname{mod}\Lambda)$.
- Morphism spaces are given by:

$$\operatorname{Hom}_{\mathcal{C}^d(\Lambda)}(X, Y) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(X, \nu_d^i Y)$$

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$$\operatorname{Hom}_{\mathcal{C}^d(\Lambda)}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(X,\nu_d^i Y)$$

- $\mathcal{C}^d(\Lambda)$ is triangulated (Keller, 2005).
- $\mathcal{C}^d(\Lambda)$ is d-Calabi-Yau, that is $\nu \simeq [d]$.
- $\operatorname{add}\Lambda$ is a vosnex d-cluster tilting subcategory of $\mathcal{C}^d(\Lambda)$

Previous Results: Cluster Categories of Hereditary Algebras

Theorem (Keller-Reiten, 2008)

Let $\mathcal{T}$ be a d-Calabi-Yau algebraic triangulated category and let $\mathcal{C} \subset \mathcal{T}$ be a vosnex d-cluster tilting subcategory. If there is a finite-dimensional hereditary algebra Λ and an equivalence $F : \operatorname{add}\Lambda \to \mathcal{C}$, then there is a triangle equivalence $\widehat{F} : \mathcal{C}^d(\Lambda) \to \mathcal{T}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}^d(\Lambda) \arrow[r, dashed, "\widehat{F}"] & \mathcal{T} \\
\operatorname{add}\Lambda \arrow[u, hook] \arrow[r, "F"'] & \mathcal{C} \arrow[u, hook]
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Previous Results: ν_2-Finite Algebras

Theorem (Amiot-Oppermann, 2012)

Let $\mathcal{T}$ be a algebraic triangulated category with a Serre functor and let $\mathcal{C} \subset \mathcal{T}$ be a 2-cluster tilting subcategory. If there is a ν_2-finite algebra Λ and an equivalence $F : \mathcal{U}_2(\Lambda) \to \mathcal{C}$ of ν_2-categories, then there is a triangle equivalence $\widehat{F} : \mathcal{D}^b(\operatorname{mod}\Lambda) \to \mathcal{T}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}^b(\operatorname{mod}\Lambda) \arrow[r, dashed, "\widehat{F}"] & \mathcal{T} \\ \mathcal{U}_2(\Lambda) \arrow[u, hook] \arrow[r, "F"'] & \mathcal{C} \arrow[u, hook] \end{tikzcd}$$

Observe that vosnexity is an empty condition when $d = 2$.

Main Theorem

Theorem (Norlén Jäderberg, 2026)

Let $\mathcal{T}$ be a algebraic triangulated category with a Serre functor and let $\mathcal{C} \subset \mathcal{T}$ be a vosnex d-cluster tilting subcategory. If there is a finite-dimensional hereditary algebra Λ and an equivalence $F : \mathcal{U}_d(\Lambda) \to \mathcal{C}$ of ν_d-categories, then there is a triangle equivalence $\widehat{F} : \mathcal{D}^b(\operatorname{mod} \Lambda) \to \mathcal{T}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}^b(\operatorname{mod}\Lambda) \arrow[r, dashed, "\widehat{F}"] & \mathcal{T} \\
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Application: Subquotients of the Derived Category

Let Λ be a hereditary algebra. Fix a primitive idempotent $e \in \Lambda$.

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Consider the full subcategory of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ consisting of the objects:

$$\mathcal{E}^d(e\Lambda) := \{X \in \mathcal{T} : \operatorname{Hom}(\mathcal{U}_d(e\Lambda), X[i]) = 0,\ 0 < i < d\}$$

By some general results due to Iyama and Yoshino (2008), the quotient category $\mathcal{E}^d(e\Lambda)/\langle \mathcal{U}_d(e\Lambda) \rangle$ is triangulated, algebraic and admits a Serre functor.

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The image of $\mathcal{U}_d(\Lambda)$ under the projection functor

$$\mathcal{E}^d(e\Lambda) \to \mathcal{E}^d(eA)/\langle\mathcal{U}_d(e\Lambda)\rangle$$

is a vosnex d-cluster tilting subcategory equivalent to $\mathcal{U}_d(\Lambda/\Lambda e\Lambda)$.

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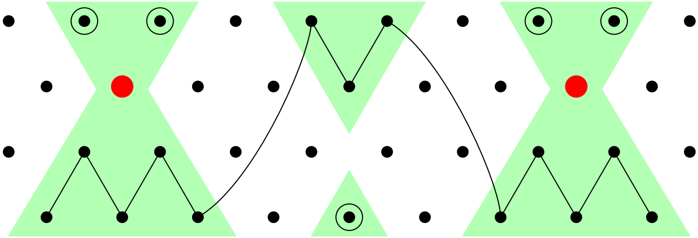
Using our results, we obtain a triangle equivalence

$$\mathcal{D}^b(\operatorname{mod}\Lambda/\Lambda e\Lambda) \cong \mathcal{E}^d(e\Lambda)/\langle\mathcal{U}_d(e\Lambda)\rangle$$

Example: A_4

Set $\Lambda = \Bbbk(1 \to 2 \to 3 \to 4)$ and $e = e_3$. For $d = 3$, the subcategory $\mathcal{E}^3(e\Lambda)$ of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is indicated by the green regions in the figure below. The red dots correspond to the ν_3-orbit $\mathcal{U}_3(e\Lambda)$ of $e\Lambda$.



$$\frac{\mathcal{E}^3(e\Lambda)}{\langle \mathcal{U}_3(e\Lambda)\rangle} \cong \mathcal{D}^b(\operatorname{mod}\Lambda/\Lambda e\Lambda) \cong \mathcal{D}^b(\operatorname{mod}\Bbbk(1 \to 2) \times \Bbbk)$$

Thank you!

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