Zigzag-like algebras and representation type of Jordan algebras

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Definition

A **Jordan algebra** is a commutative k-algebra $(J, \cdot)$ satisfying

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x) \quad x, y \in J.$$

Any associative algebra A gives rise to a Jordan algebra $(A^+, \cdot)$

$$x \cdot y = xy + yx.$$

A Jordan algebra is called **special** if it can be realized as a Jordan subalgebra of some A^+. The subspace of Hermitian elements $H(A, \sigma) = \{\sigma(x) = x \,|\, x \in A\}$ forms a Jordan subalgebra of A^+.

Jordan algebra **of Clifford type** is $J(f, n) = \mathrm{k}1 \oplus \mathrm{k}^n$, $n \geq 2$, where $v \cdot w = f(v, w)1$, $v, w \in \mathrm{k}^n$, f is a symmetric bilinear form on k^n, and 1 acts as a unit.

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The formal definition of a **(bi)module** over a Jordan algebra J due to Eilenberg says that it is a vector space M endowed with a mapping $a \to \rho(a)$ of J to the associative algebra $\mathrm{End}(M)$ such that the split extension algebra $J \oplus M$ is Jordan algebra as well.

Another source of mappings of J into $\mathrm{End}(M)$ are Jordan homomorphisms $\sigma : J \to \mathrm{End}(M)^+$. It is straightforward to check that σ define a structure of J-module on M. We call such modules **special**.

As usual, the category of J-modules (resp. the category of special J-modules) is equivalent to the category left modules over the **universal multiplicative enveloping algebra** $U = U(J)$ (resp. the **universal special enveloping algebra** $S = S(J)$).

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1. If $\dim_k J < \infty \ \Rightarrow \dim_k U(J) < \infty$, $\dim_k S(J) < \infty$
2. If $\dim_k J < \infty$ and J is simple $\Rightarrow U(J)$, $S(J)$ are semi-simple.
3. If $e \in J$ is unit element then

$$\begin{array}{ccccccc} & J\text{-mod} \simeq & J\text{-mod}_0 & \oplus & J\text{-mod}_{\frac{1}{2}} & \oplus & J\text{-mod}_1, \\ e \text{ acts as} & & 0 & & \frac{1}{2} & & 1 \end{array}$$

   J-mod$_1$ category of **unital** modules, $J\text{-mod}_{\frac{1}{2}} \simeq S(J)$-mod, where $S(J) = T(J)/\langle ab + ba - 2a \cdot b \rangle$, category of **special** modules.
4. For J simple Jacobson described unital and special modules. In particular, if J is simple then either $S(J)$ is simple or $S(J)$ is direct sum of two copies of matrix algebra.
5. For $J_1 \oplus J_2$ simple modules are $M_i \in J_i$-mod$_1$, $i = 1, 2$ and $V_1 \otimes V_2$, where $V_i \in J_i\text{-mod}_{\frac{1}{2}}$.

Conclusion: Special representations of non-simple Jordan algebra $J = J_s + \operatorname{Rad} J$ are representations of finite-dimensional associative algebra $S(J) = S(J_s) + \operatorname{Rad} S(J)$.

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Drozd's trichotomy theorem

Let $\mathcal{C}$ be a module category. Then exactly one of the following holds

- $\mathcal{C}$ **has finite representation type:** the indecomposables are classified by a few discrete parameters;
- $\mathcal{C}$ **has tame representation type:** the indecomposables are classified by finitely many discrete parameters and one continuous sub-parameter;
- $\mathcal{C}$ **has wild representation type:** any classification theorem would involve simultaneously classifying the modules over every other ring as well.

To determine a type of $\mathcal{C}$ we associate to it a quiver $Q = (Q_0, Q_1)$

$$Q_0 = \{\text{simple modules } L_1, \dots, L_r \text{ in } \mathcal{C}\}$$

$$Q_1 = \{\#\text{arrows from vertex } L_i \text{ to vertex } L_j \text{ is } \dim \operatorname{Ext}^1(L_j, L_i)\}$$

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Let Q be a quiver, the **quiver double** $d(Q)$ of Q is:

$$d(Q_0) = \{X^+, X^- \mid X \in Q_0\}$$
$$d(Q_1) = \{\tilde{a} : s(a)^- \to e(a)^+ \mid a \in Q_1\}.$$

Theorem (Gabriel)

Let A be a finite dimensional associative algebra over algebraically closed field, such that $\mathrm{Rad}^2 A = 0$, Q its quiver. Then A is of finite (tame) representation type if and only if $d(Q)$ is a disjoint union of simply-laced Dynkin diagrams (extended Dynkin diagrams).

Describe special representations for finite-dimensional Jordan algebra $J = J_s \oplus \mathrm{Rad}\, J$ with $\mathrm{Rad}^2\, J = 0$.

Albert classification of simple finite-dimensional Jordan algebras:

- Jordan algebra of Clifford type $J(f, n)$;
- matrix algebras $H_n(D)$, $n \geq 3$, D composition algebra of dimension 1, 2, 4 or 8 for $n = 3$;
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Theorem (K-Ovsienko-Shestakov, 2011)

Let J be a finite dimensional Jordan algebra of matrix type, $\operatorname{Rad}^2 J = 0$. Then $\operatorname{Rad}^2 S(J) = 0$. Thus J is of one-sided representation finite (respectively tame) type, if and only if $d(Q(J))$ is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).

When Jordan algebra is not of matrix type $\operatorname{Rad}^2 S(J) \neq 0$.

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Let $\mathfrak{T}$ be the category of $\mathfrak{sl}(2)$-modules M such that

$$M = L(0) \otimes A \oplus L(2) \otimes B$$

where $L(0)$ is the trivial $\mathfrak{sl}(2)$-module, $L(2)$ is the adjoint $\mathfrak{sl}(2)$-module, while A and B are vector spaces.

Tits noted that for each Lie algebra $\mathfrak{g}$ in $\mathfrak{T}$ one can define a Jordan algebra structure on the vector space of weight 2 elements in $\mathfrak{g}$

$$x \cdot y := \frac{1}{2}[x, f(y)],$$

where $f(-)$ refers to the action of $f \in \mathfrak{sl}(2)$ on $\mathfrak{g}$. This functor from the category $\mathfrak{T}$ to the category of Jordan algebras is called the **Tits functor.**

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One more description for J-mod$_{\frac{1}{2}}$

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Tits-Kantor-Koecher construction for Jordan algebra

For any $x, y \in J$ the map

$$\partial_{x,y} : z \mapsto x \cdot (z \cdot y) - (x \cdot z) \cdot y$$

is a derivation in J. To a Jordan algebra J, one can associate the object

$$L(0) \otimes \mathrm{Inner}(J) \oplus L(2) \otimes J \in \mathfrak{T},$$

equipped with a Lie algebra structure

1. for two elements of $L(0) \otimes \mathrm{Inner}(J) \cong \mathrm{Inner}(J)$, their Lie bracket is the Lie bracket of derivations;
2. for an element of $\mathrm{Inner}(J)$ and an element of $L(2) \otimes J$, their Lie bracket comes from the action of $\mathrm{Inner}(J)$ on $L(2) \otimes J$;
3. for $u_1 \otimes z_1, u_2 \otimes z_2 \in L(2) \otimes J$, one defines

$$[u_1 \otimes z_1, u_2 \otimes z_2] := K(u_1, u_2)\partial_{z_1,z_2} + [u_1, u_2] \otimes (z_1 \cdot z_2),$$

where $K(-,-)$ is the Killing form on $\mathfrak{sl}_2$.

Relations between J-mod and $\mathfrak{g} = TKK(J)$-modules?
We define two adjoint functors *Jor* and *Lie* between J-mod and $\mathfrak{g}$-modules admitting a short grading.

Not every J-module can be obtained from a $\mathfrak{g}$-module by application of *Jor*: one has to consider $\hat{\mathfrak{g}}$ the universal central extension of $\mathfrak{g}$.

Let $\mathcal{S}$ (resp. $\mathcal{S}_{\frac{1}{2}}$) be the category of $\hat{\mathfrak{g}}$-modules M such that the action of α_J induces a short grading on M (resp. a grading of length 2, namely $M_{-1} \oplus M_1$).

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J	$\mathfrak{g}$	$\mathcal{S}_{\frac{1}{2}}$	$\mathcal{S}$
$H_n(D_1)$	$\mathfrak{sp}(2n)$	V	$ad,\ \Lambda^2 V$
$H_n(D_2)$	$\mathfrak{sl}(2n)$	$V,\ V^*$	$ad,\ S^2(V),\ S^2(V^*),$ $\Lambda^2(V),\ \Lambda^2(V^*)$
$H_n(D_3)$	$\mathfrak{so}(4n)$	V	$ad,\ S^2(V) \quad *$
$J(f,n)$ $n=2\nu$	$\mathfrak{so}(n+3)$	Γ spinor	$\Lambda^i(V),\ i=1,\dots,\nu+1$
$J(f,n)$ $n=2\nu-1$	$\mathfrak{so}(n+3)$	$\Gamma^+,\ \Gamma^-$ spinor	$\Lambda^i(V),\ i=1,\dots,\nu$ $\Lambda^{\nu+1}(V)^{\pm}$
$\mathbb{A}$	E_7		ad

Category $\mathcal{S}_{\frac{1}{2}}$

Lemma (K-Serganova, 2016)

Let $\mathfrak{g} = \mathfrak{g}_s + \mathfrak{r}$ be the Levi decomposition of $\mathfrak{g}$, $\mathfrak{r}^2 = 0$. Let L and L' be two simple $\mathfrak{g}_s$-modules in $\mathcal{S}_{\frac{1}{2}}$, then $\dim \operatorname{Ext}^1(L, L')$ equals the multiplicity of L' in $L \otimes \mathfrak{r}$.

If $\mathfrak{r}^{(q)} = L_1 \otimes L_2$ where L_1 and L_2 are simple $\mathfrak{g}_s^{(i)}$ and $\mathfrak{g}_s^{(j)}$-modules:

$$\begin{tikzcd} {[L_1]} \arrow[dr] & {[L_2]} \arrow[dl] \\ {[L_1^*]} & {[L_2^*]} \end{tikzcd}$$

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Proposition ($\alpha\beta \neq 0$, K-Serganova, 2023)

Let α and β be two arrows in Q corresponding to $\mathfrak{r}^{(p)}$ and $\mathfrak{r}^{(q)}$ respectively. If $\alpha\beta \neq 0$,

- *then $\mathfrak{r}^{(p)} \simeq (\mathfrak{r}^{(q)})^*$ as $\mathfrak{g}_s$-modules. Moreover, in the case $p = q$ the invariant pairing $\mathfrak{r}^{(p)} \times \mathfrak{r}^{(p)} \to \mathrm{k}$ is skew-symmetric.*
- *then the head of β coincides with the tail of α. In other words the corresponding path is a cycle.*
- *ether $\mathfrak{r}^{(q)} = L \otimes S$, where L is the standard module over a simple ideal isomorphic to $\mathfrak{sl}_2$ and S is some simple $\mathfrak{g}_s$-module with grading of length 2.*
- *or $\mathfrak{g}_s = \mathfrak{so}_m$ with the second short grading and $\mathfrak{r}^{(q)} = \mathfrak{r}^{(p)} = V$, where V is the standard $\mathfrak{so}_m$-module.*

Theorem (K-Serganova, 2024)

The category J-mod$_{\frac{1}{2}}$ is equivalent to the category of representations of finite-dimensional graded quadratic algebra $\mathcal{A}$. Moreover, $\mathcal{A}$ is Koszul.

Outcome: To determine which quivers are finite and tame one has to:

- determine quivers which satisfy Gabriel's criterium for $\mathrm{Rad}^2 = 0$;
- label some of the vertices corresponding to standard representation of $\mathfrak{sl}_2$ and add some non-zero cyclic relations.
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In general to determine quivers which satisfy Gabriel's criterium for $\mathrm{Rad}^2 = 0$ is combinatorially extremely difficult.

However $S(J)$ has an involution trivial on J and establishes duality functor in J-$\mathrm{mod}_{\frac{1}{2}}$.

$J_s \in \{J(f, n),\ n \text{ even};\ H_n(D_1),\ H_n(D_3), \mathrm{k}\}$ $S(J_s)$ is simple;

$J_s \in \{J(f, n),\ n \text{ odd};\ H_n(D_2)\}$ its $S(J_s)$ is sum of two simple.

Example ($J_i \in \{J(f, n),\ n$ even; $H_n(D_1),\ H_n(D_3), \mathrm{k}\}$)

$J_1 + J_2 + J_3 + M_1 + M_2 + M_3 + V_1 \otimes V_2 + V_2 \otimes V_3$

$$Q: \quad \begin{tikzcd} V_1 \arrow[loop above] \arrow[r, shift left] & V_2 \arrow[loop above] \arrow[l, shift left] \arrow[r, shift left] & V_3 \arrow[loop above] \arrow[l, shift left] \end{tikzcd}$$

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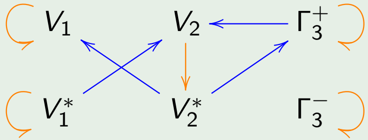
$\mathfrak{g} \in \{\mathfrak{sl}(2n), \mathfrak{so}^2(2m)\}$

Example

$$\mathfrak{sl}(2n) + \mathfrak{sl}(2m) + \mathfrak{so}(8) + ad_1 + S^2 V_2^* + \Lambda^2 V_3 + \\ + V_1 \otimes V_2 + V_2 \otimes \Gamma_3^+$$

Q :



$$\mathfrak{sl}(2n) + \mathfrak{so}(8) + \mathfrak{so}(8) + S^2 V_1^* + \Lambda_2^+ + \Lambda_3^- + V_1^* \otimes \Gamma_2^- + \Gamma_2^- \otimes \Gamma_3^+$$

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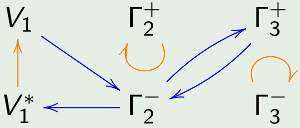

To answer the first question we have "to comb" $Q = Q(S(J))$:

$$Q \xrightarrow{graph\ \ of\ \ J} \Gamma \quad \xrightarrow{double\ \ quiver} D(\Gamma)$$

Graph of Jordan algebra

We introduce the graph $\Gamma = \Gamma(J) = \Gamma(\mathfrak{g})$, where

$$\mathfrak{g} = \mathfrak{g}_s + \mathfrak{r} = \bigoplus_{i=1}^{k} \mathfrak{g}_s^{(i)} + \bigoplus_{j=1}^{m} \mathfrak{r}^{(j)}.$$

- The set of vertices Γ_0 coincides with Q_0.
- Every j produces several edges of Γ by the rule $V - W$ if $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{(j)}, V \otimes W) = \mathrm{k}$.

Double quiver

For a (undirected) graph Γ we define the **double quiver** $D(\Gamma) := (D(\Gamma)_0, D(\Gamma)_1, \mathfrak{s}, \mathfrak{t})$ of Γ by

- $D(\Gamma)_0 := \Gamma_0$;
- $D(\Gamma)_1 := \{\alpha_{a,b}, \alpha_{b,a} \mid \alpha \in \Gamma_1, \tau(\alpha) = \{a, b\}, a \neq b\} \bigcup \{\alpha_{a,a} \mid \alpha \in \Gamma_1, \tau(\alpha) = \{a, a\}\}$, $\mathfrak{s}(\alpha_{a,b}) := a, \quad \mathfrak{t}(\alpha_{a,b}) := b$.

There is an obvious involution on $D(\Gamma)$: $\sigma_{D(\Gamma)}(\alpha_{a,b}) = \alpha_{b,a}$.

Let Γ be a graph, and let $D(\Gamma)$ be its double quiver. Denote by $\mathrm{k}\Gamma$ the path algebra of the quiver $D(\Gamma)$ with coefficients in k and let $\mathcal{I}$ be an admissible ideal in $\mathrm{k}\Gamma$. We call $\Lambda(\Gamma, \mathcal{I}) := \mathrm{k}\Gamma/\mathcal{I}$ the **graph algebra** of the pair $(\Gamma, \mathcal{I})$.

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Example

$$\mathfrak{sl}(2n) + \mathfrak{sl}(2m) + \mathfrak{so}(8) + ad_1 + S^2 V_2^* + \Lambda^2 V_3 + V_1 \otimes V_2 + V_2 \otimes \Gamma_3^+$$

$$Q \to \Gamma \to D(\Gamma)$$

$$\begin{tikzcd}
V_1 \arrow[loop left] & V_2 \arrow[d] & \Gamma_3^+ \arrow[l] \arrow[loop right] \\
V_1^* \arrow[loop left] \arrow[ur] & V_2^* \arrow[ul] \arrow[ur] & \Gamma_3^- \arrow[loop right]
\end{tikzcd}
\qquad
\begin{tikzcd}
V_1 \arrow[r, dash] \arrow[d, dash] & V_2 \arrow[r, dash] & \Gamma_3^+ \arrow[d, dash] \\
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\end{tikzcd}$$

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\end{tikzcd}$$

Example (Combing the quiver)

$$\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}^1_{2m} \oplus \mathfrak{sl}^2_{2k} \oplus \mathfrak{sl}^3_{2l} \oplus \mathfrak{so}^4_{8n+1} \ltimes \Gamma_4 \otimes V \oplus V \otimes V_1 \oplus$$
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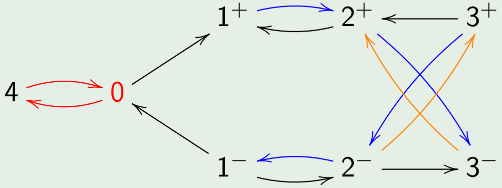

Associating the graph Γ

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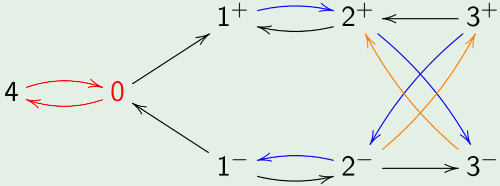

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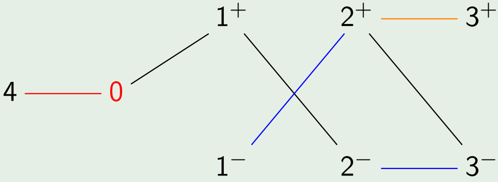

Example (Combed quiver)

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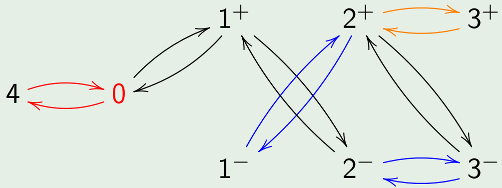

Proposition

$$D(\Gamma)_0 = Q_0 \qquad D(\Gamma)_1 = \{L_1 \to L^* \mid L_1 \to L \in Q_1\}.$$

Denote by $\mathcal{A} = \mathrm{k}Q/\mathcal{I}$ the basic algebra which correspond to J-mod$_{\frac{1}{2}}$ and by $\Lambda(\Gamma, \mathcal{I})$ the graph algebra obtained from Γ.

Theorem (Bekkert-K-Serganova, 2025)

1. *$\mathcal{A}$ is of finite type $\Longleftrightarrow \Lambda(\Gamma, \mathcal{I})$ is of finite type.*
2. *$\mathcal{A}$ is of tame type $\Longleftrightarrow \Lambda(\Gamma, \mathcal{I})$ is of tame type.*

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Zigzag and zigzag-like algebras

In 2001 Huertano, Khovanov introduced **zigzag algebras** $A(\Gamma)$. Let Γ be a connected graph without loops and multiple edges. Denote by $(a|b|c)$ a path that starts in a, goes to b and then to c. Then $A(\Gamma)$ the quotient algebra of $D(\Gamma)$ by the ideal generated:

- $(a|b|c)$ for each $a, b, c \in \Gamma$ with $a \neq c$.
- $(a|b|a) - (a|c|a)$, when a is connected to both b and c.

Let Γ be a graph and $D(\Gamma)$ be the double quiver of Γ. Denote by

$$I_{zz} := I_{zz}(D(\Gamma)),$$

the ideal generated by $\beta\alpha$, where $\beta, \alpha \in D(\Gamma)$ and $\beta \neq \sigma_{D(\Gamma)}(\alpha)$.

We call the graph algebra $\Lambda(\Gamma, I)$ a **zigzag-like** algebra if $I_{zz} \subseteq I$.

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Bekkert, K., Serganova 2025: obtained description of finite and tame zigzag-like algebras in terms of generalized Dynkin and Euclidean diagrams with labels.

Theorem (Bekkert, K., Serganova 2025)

Let J be a finite-dimensional Jordan algebra with radical square zero and $\Lambda(\Gamma, \mathcal{I})$ be the corresponding graph algebra. Then $\Lambda(\Gamma, \mathcal{I})$ is a zigzag-like algebra.

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Theorem (Bekkert, K., Serganova 2025)

Let J be a finite-dimensional Jordan algebra with radical square zero, and let $\Lambda(\Gamma, \mathcal{I})$ be the corresponding graph algebra. TFAE:

1. *J is of finite representation type;*
2. *Γ is a disjoint union of simply laced generalized Dynkin diagrams and does not contain the following subgraphs:*

$$(i): \quad \circ \overset{\alpha}{\text{—}} a_1 \text{—} \cdots \text{—} a_n \overset{\beta}{\text{—}} \circ, \quad (a_1, \alpha), (a_n, \beta) \in \Omega(\Lambda), n \geq 1.$$

$$(ii): \quad \begin{tikzcd}[arrows=dash] & \circ \arrow[d] & & & & \\ \circ \arrow[r] & a_1 \arrow[r] & a_2 \arrow[r] & \cdots \arrow[r] & a_n \arrow[r, "\alpha"] & \circ \end{tikzcd}, \quad (a_n, \alpha) \in \Omega(\Lambda), n \geq 1.$$

$$(iii): \quad \begin{tikzcd}[arrows=dash] a_1 \arrow[loop left] \arrow[r] & a_2 \arrow[r] & \cdots \arrow[r] & a_n \arrow[r, "\alpha"] & \circ \end{tikzcd}, \quad (a_n, \alpha) \in \Omega(\Lambda), n \geq 1.$$

Theorem (Bekkert, K., Serganova, 2025)

1. *J is tame representation type;*
2. *Γ is a disjoint union of simply laced generalized Dynkin or Euclidean diagrams and does not contain the following subgraphs:*

$$\begin{array}{ccccc} & & \circ & & \\ & & | & & \\ \circ & \overset{\alpha}{—} & a & \overset{\beta}{—} & \circ \end{array} \quad \textit{and} \quad \begin{array}{ccccc} & & \circ & & \\ & & | & & \\ \circ & — & a & \overset{\alpha}{—} & \circ, \\ & & | & & \\ & & \circ & & \end{array} \quad (a,\alpha),(a,\beta) \in \Omega(\Lambda).$$

$$\begin{array}{ccccccccccc} & & & & \circ & & & & & & \\ & & & & | & & & & & & \\ \circ & — & \circ & — & a_1 & — & \cdots & — & a_n & \overset{\alpha}{—} & \circ, \end{array} \quad (a_n,\alpha) \in \Omega(\Lambda), n \geq 1.$$

$$\begin{array}{ccccccc} & & \circ & & & & \\ & & |\,\alpha & & & & \\ \circ & — & a_1 & — & \cdots & — & a_n \circlearrowright, \end{array} \quad (a_1,\alpha) \in \Omega(\Lambda), n \geq 1.$$