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Compatible Lie algebras and their representations

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MOTIVATION

One of the two classes of nonlinear hyperbolic system of partial differential equations considered in the present paper consists of equations of the form

$$u_x = [u, v], \qquad v_y = [v, u]_1, \tag{1}$$

where u and v belong to a vector space $\mathscr{G}$ equipped with two Lie brackets $[\cdot\,,\cdot]$ and $[\cdot\,,\cdot]_1$. For the well-known integrable principal chiral model

$$u_x = [u, v], \qquad v_y = [u, v], \tag{2}$$

the brackets $[\cdot\,,\cdot]$ and $[\cdot\,,\cdot]_1$ coincide neglecting the sign.

It turns out that if the Lie algebra with bracket $[\cdot\,,\cdot]$ is semisimple and the second bracket is compatible with the first, then equation (1) is integrable.

+ CONNECTIONS WITH YANG-BAXTER EQ.

HOMOGENEOUS SUB ALGEBRAS OF LOOP ALGEBRAS OF SIMPLE LIE ALGEBRAS

Compatibility

Two algebraic structures of the same type $(V, *_1)$ and $(V, *_2)$ with the same underlying vector space are said to be compatible if any linear combination of $*_1$ and $*_2$ is again a product of the same type.

Let $\underline{\mathfrak{g}} = (\mathfrak{g}, [-,-])$ and $\underset{\sim}{\mathfrak{g}} = (\mathfrak{g}, \{-,-\})$ be two Lie algebras over a field $\mathbb{K}$ defined on the same vector space $\mathfrak{g}$. Then the following conditions are equivalent:

- *$(\mathfrak{g}, [\![-,-]\!]_{\lambda,\lambda'})$ is a Lie algebra for all $\lambda, \lambda' \in \mathbb{K}$, where $[\![x,y]\!]_{\lambda,\lambda'} = \lambda [x,y] + \lambda' \{x,y\}$ for all $x, y \in \mathfrak{g}$;*
- *$(\mathfrak{g}, [\![-,-]\!])$ is a Lie algebra, where $[\![x,y]\!] = [x,y] + \{x,y\}$ for all $x, y \in \mathfrak{g}$;*
- *The following identity (named the mixed Jacobi identity) holds for all $x, y, z \in \mathfrak{g}$:*

$$\{[x,y], z\} + \{[y,z], x\} + \{[z,x], y\} + [\{x,y\}, z] + [\{y,z\}, x] + [\{z,x\}, y] = 0.$$

→ CLA

Definition

A *compatible Lie algebra* is a triple $(\mathfrak{g}, [-,-], \{-,-\})$, where $\underline{\mathfrak{g}} = (\mathfrak{g}, [-,-])$ and $\underset{\sim}{\mathfrak{g}} = (\mathfrak{g}, \{-,-\})$ are Lie algebras satisfying any of the the previous three equivalent conditions.

NOTES:

1) IF $\{\cdot,\cdot\} = \lambda [\cdot,\cdot]$ THEN $(\mathfrak{g}, [\,,\,], \{\,\})$ IS COMPATIBLE

2) IF $(\mathfrak{g}, [\,,\,], \{\,\})$ IS COMPATIBLE AND $\varphi \in \mathrm{Aut}(\mathfrak{g}, [\,,\,])$ ($= \underline{\mathfrak{g}}$)

THEN $(\mathfrak{g}, [\,,\,], \{\,\}')$ IS COMPATIBLE, WHERE

$\{x,y\}' = \varphi^{-1}(\{\varphi(x), \varphi(y)\}) \quad \forall\, x,y \in \mathfrak{g}.$

The mixed Jacobi identity

$$\{[x,y],z\}+\{[y,z],x\}+\{[z,x],y+[\{x,y\},z]+[\{y,z\},x]+[\{z,x\},y]=0.$$

is equivalent to $\{\,,\,\} \in Z^2(\underline{\mathfrak{g}},\underline{\mathfrak{g}})$.

Thus, compatible structures on $\underline{\mathfrak{g}}$ $\longleftrightarrow$ 2-COCYCLES ON $Z^2(\underline{\mathfrak{g}},\underline{\mathfrak{g}})$ SATISFYING THE JACOBI IDENTITY.

In particular, by the 2nd Whitehead Lemma, if $\mathfrak{g}$ is finite-dim'l semisimple (char $\mathbb{F}=0$) then $H^2(\mathfrak{g},\mathfrak{g})=0$

so

$$\{a,b\} = [\alpha(a),b]+[a,\alpha(b)]-\alpha([a,b])$$

$\forall a,b\in\mathfrak{g}$

for some $\alpha\in End_{v.s.}(\mathfrak{g})$.

EXAMPLES

Example (A not so trivial example)

Let $\mathfrak{g}$ be a three-dimensional vector space generated by x, y, z. Define the following products:

HEISENBERG

$$[x, y] = z, \qquad \text{and}$$

$$\{x, y\} = z, \qquad \{z, x\} = 2x, \qquad \{z, y\} = -2y.$$

sl_2

We check the following

$$\begin{aligned}\{[x,y],z\} + \{[y,z],x\} + \{[z,x],y\} &= \{z,z\} + \{0,x\} + \{0,y\}\\ &= 0;\\ [\{x,y\},z] + [\{y,z\},x] + [\{z,x\},y] &= [z,z] + [2y,x] + [2x,y]\\ &= 0 - 2z + 2z = 0.\end{aligned}$$

The mixed Jacobi identity, being the sum of the two expressions above, is equal to zero.

A NONTRIVIAL DOUBLE $\mathfrak{sl}_2$ EXAMPLE

ON THE BASIS $\{e, f, h\}$:

$$[e,f]=h\ , \quad [h,e]=2e\ , \quad [h,f]=-2f$$

$$\{e,f\}=-3f\ , \quad \{h,e\}=-3h\ , \quad \{h,f\}=4e$$

($\frac{h}{2}$, $\frac{f}{3}$, $\frac{2}{3}e$ IS AN $\mathfrak{sl}_2$-TRIPLE)

Center, subalgebras, ideals

Definition

The *centre* of a compatible Lie algebra $\mathfrak{g}$, denoted by $Z(\mathfrak{g})$, is the ideal defined by

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 = \{x, y\} \quad \forall y \in \mathfrak{g}\}$$

$$Z(\mathfrak{g}) = Z(\underline{\mathfrak{g}}) \cap Z(\underset{\sim}{\mathfrak{g}}).$$

Definition

A *subalgebra* of a compatible Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is closed for both products.

An *ideal* $\mathfrak{i}$ of a compatible Lie algebra $\mathfrak{g}$ is a vector subspace such that

$$[\mathfrak{i}, \mathfrak{g}], \{\mathfrak{i}, \mathfrak{g}\} \subseteq \mathfrak{i}$$

$[\![i, g]\!]_{\lambda_1, \lambda_2} \subseteq i$

$\forall \lambda_1, \lambda_2 \in \mathbb{F}$

- Kernels of homomorphisms are ideals of the domain;
- Images of homomorphisms are subalgebras of the codomain;
- Quotients are well defined;
- The usual isomorphism theorems hold.

Weyl & Levi are

incompatible

Solvable and Semisimple (compatible) Lie algebras

Recall the commutator of sublagebras

$$[\![\mathfrak{s}, \mathfrak{t}]\!] = \text{span}_{\mathbb{K}} \langle [s, t], \{s, t\} \mid s \in \mathfrak{s}, t \in \mathfrak{t} \rangle . = [\mathfrak{s},\mathfrak{t}] + \{\mathfrak{s},\mathfrak{t}\}$$

We may define the *derived series*

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(i)} \supseteq \cdots ,$$

where

$$\mathfrak{g}^{(0)} := \mathfrak{g} \text{ and } \mathfrak{g}^{(i+1)} = [\![\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]\!] .$$

Each term of this series is an ideal of the previous one (but not necessarily of $\mathfrak{g}$) and each quotient is abelian.

Definition

A compatible Lie algebra is said to be *solvable* if $\mathfrak{g}^{(i)} = 0$ for some $i \in \mathbb{N}$.

Definition

Let $\mathfrak{g}$ be a finite dimensional compatible Lie algebra. Its largest solvable ideal is called its *radical* and is denoted by $\mathrm{rad}(\mathfrak{g})$.

Definition

We say that a compatible Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathrm{rad}(\mathfrak{g}) = \{0\}$.

Remark

- A simple compatible Lie algebra is semisimple;
- For any compatible Lie algebra $\mathfrak{g}$, the compatible Lie algebra $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.

Suppose the base field has characteristic 0.

Theorem (Levi's Theorem)

Every Lie algebra $\mathfrak{g}$ is the semidirect product of a solvable ideal and a semisimple subalgebra

$$\mathfrak{g} \simeq \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g}),$$

where $\mathfrak{s} \simeq \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$.

LEVI IS NOT COMPATIBLE

Let $\mathfrak{g}$ be the compatible Lie algebra of dimension 3 defined by the following relations on the basis $\{x, y, z\}$:

$$[x, y] = x + z, \quad [y, z] = -z,$$
$$\{x, y\} = y, \quad \{x, z\} = z.$$

We have $\mathrm{rad}(\mathfrak{g}) = \mathbb{C}z$ and $\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \simeq CL_{2,4}$.

But $\mathfrak{g}$ has no subalgebra isomorphic to $CL_{2,4}$, so Levi's Theorem fails!

$CL_{2,4}$: $\mathbb{F}x \oplus \mathbb{F}y$

$[x, y] = x, \quad \{x, y\} = y$

SS $\not\Rightarrow$ $\oplus$ simples

Let $\mathfrak{g}$ be the compatible Lie algebra of dimension 3 defined by the following relations on the basis $\{x, y, z\}$:

$$[x, y] = x, \qquad [x, z] = x, \qquad [y, z] = x,$$
$$\{x, y\} = y, \qquad \{x, z\} = y, \qquad \{y, z\} = y.$$

This algebra has a single nontrivial ideal isomorphic to $CL_{2,4}$.

It is thus semisimple but it cannot be decomposed into a direct sum of simple ideals.

$$0 \longrightarrow CL_{2,4} \hookrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/CL_{2,4} \longrightarrow 0$$

Unique nontriv. ideal of $\mathfrak{g}$

$\mathrm{rad}(\mathfrak{g}) = 0$

$\mathfrak{g}$ is not $\oplus$ of simples although it is SS. (and not simple).

Cohomology has also been defined for CLAs and it has the usual interpretations in low degrees.

However, the Whitehead lemmas also fail for simple CLAs.

Representations

Definition

A *representation* of a compatible Lie algebra $\mathfrak{g}$ is a triple (V, ρ, μ), where

- (V, ρ) is a representation of $(\mathfrak{g}, [-,-])$,
- (V, μ) is a representation of $(\mathfrak{g}, \{-,-\})$, and
- $(V, \rho + \mu)$ is a representation of $(\mathfrak{g}, [\![-,-]\!])$.

In other words,

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

$$\mu(\{x,y\}) = \mu(x)\mu(y) - \mu(y)\mu(x).$$

$$\rho(\{x,y\}) + \mu([x,y]) = \rho(x)\mu(y) - \mu(y)\rho(x) + \mu(x)\rho(y) - \rho(y)\mu(x).$$

$$= [\rho(x), \mu(y)] + [\mu(x), \rho(y)] \quad \text{in } \mathfrak{gl}(V).$$

Let $CL_{2,4}$ be the compatible Lie algebra of dimension 2 with basis elements x and y and products

$$[x, y] = x, \quad \{x, y\} = y.$$

It is the smallest simple compatible Lie algebra.

It is a counterexample to Weyl's theorem!

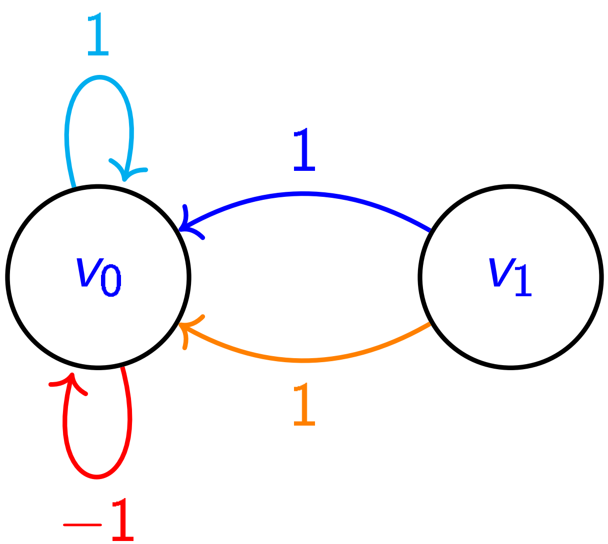
ALTHOUGH WE HAD SEEN ONE ALREADY AS $(\mathfrak{g}, \underline{ad}, \underset{\sim}{ad})$ IS A REPRESENTATION OF $(\mathfrak{g}, [\,,\,], \{\,,\,\})$.

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mu(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mu(y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

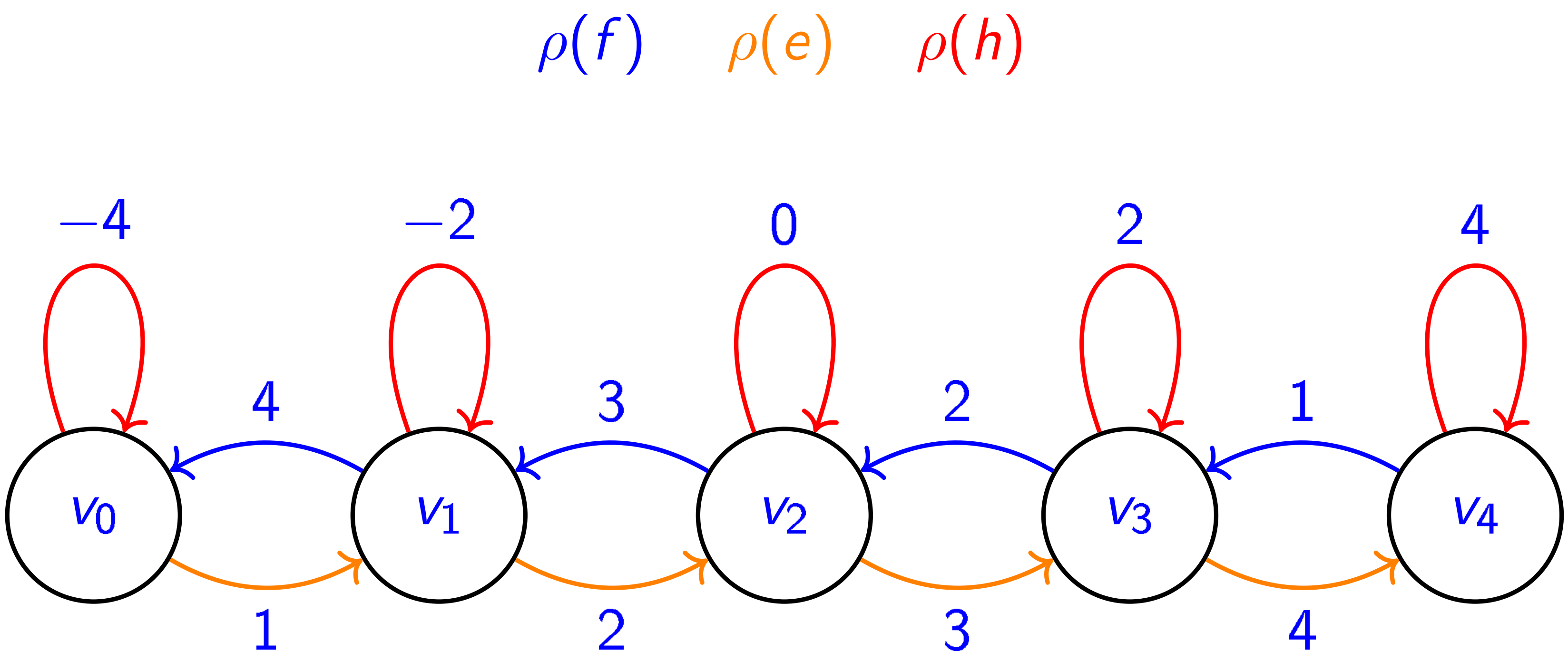
Compatible Lie algebras are interesting

$\rho(x)$ $\rho(y)$ $\mu(x)$ $\mu(y)$



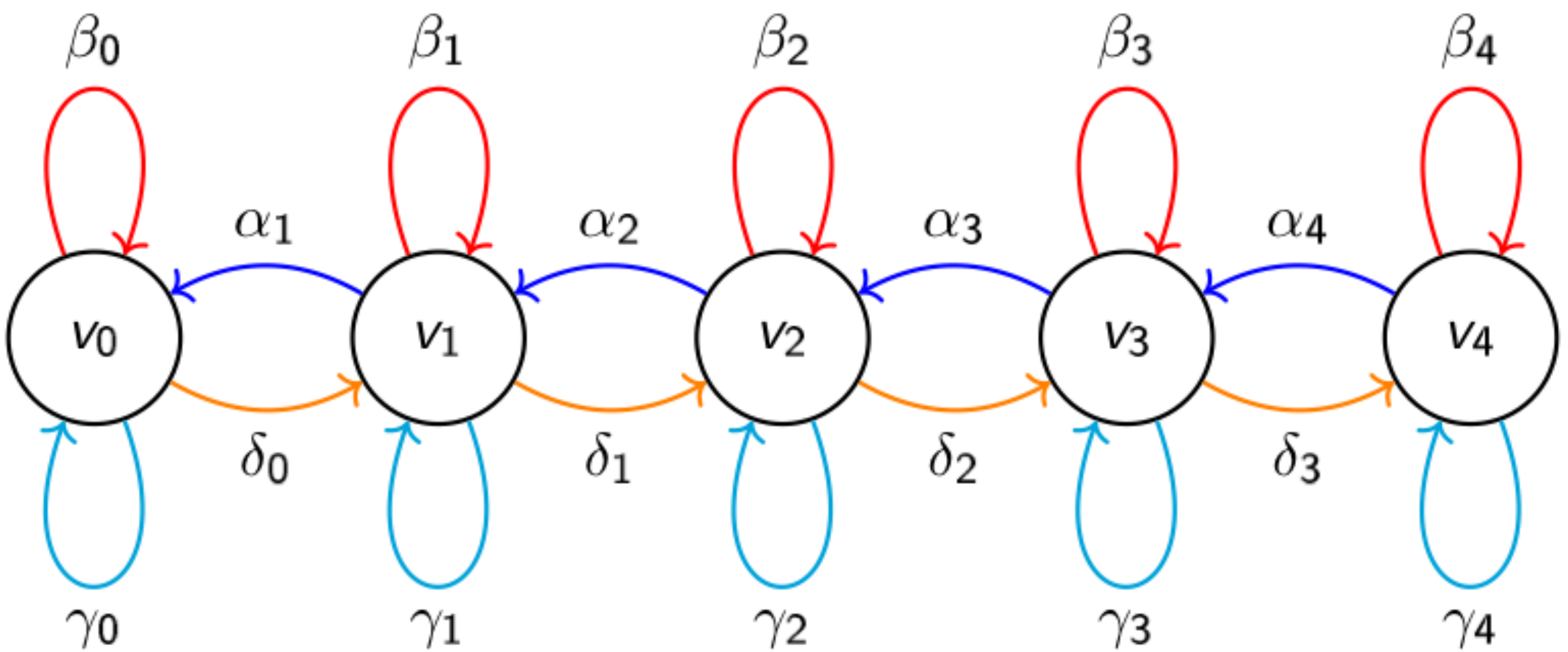
INDECOMPOSABLE BUT NOT SIMPLE.
SO WEYL'S THEOREM FAILS.

Finite-dimensional representations of $\mathfrak{sl}_2$



Line representations of $CL_{2,4}$

$\rho(x)$ $\rho(y)$ $\mu(x)$ $\mu(y)$



$\alpha_i, \beta_i, \delta_i, \gamma_i \in \mathbb{F}$

Theorem

Let V be an irreducible finite-dimensional line representation of $CL_{2,4}$ of dimension $n+1$. Then the coefficients α_i, β_i, δ_i and γ_i satisfy the following:

$$\alpha_{i+1}\delta_i = (i+1)(i-n), \quad \beta_i = \beta_0 + i, \quad \beta_i + \gamma_i = -n + 2i.$$

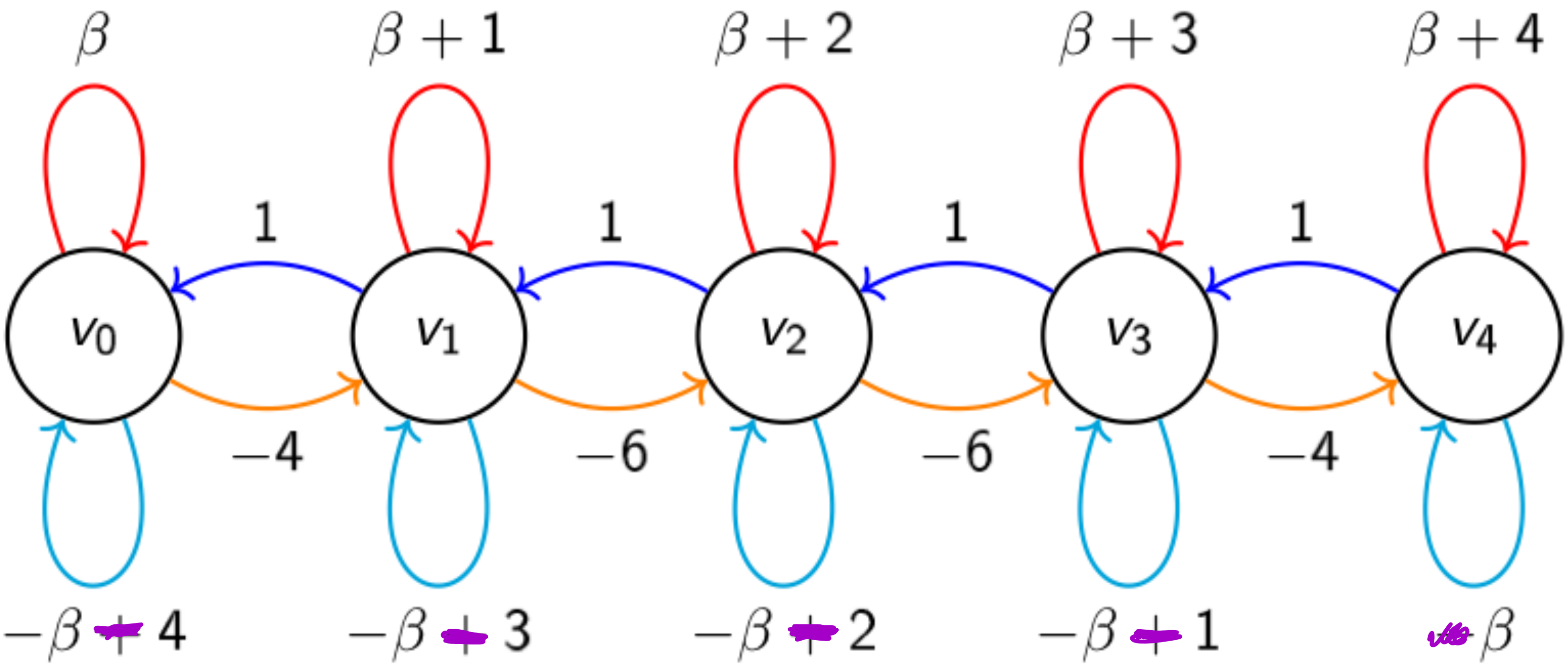
Moreover, the isomorphism class only depends on the value of β_0.

We name each of these isomorphism classes $V(n, \beta)$, $\beta \in \mathbb{F}$.

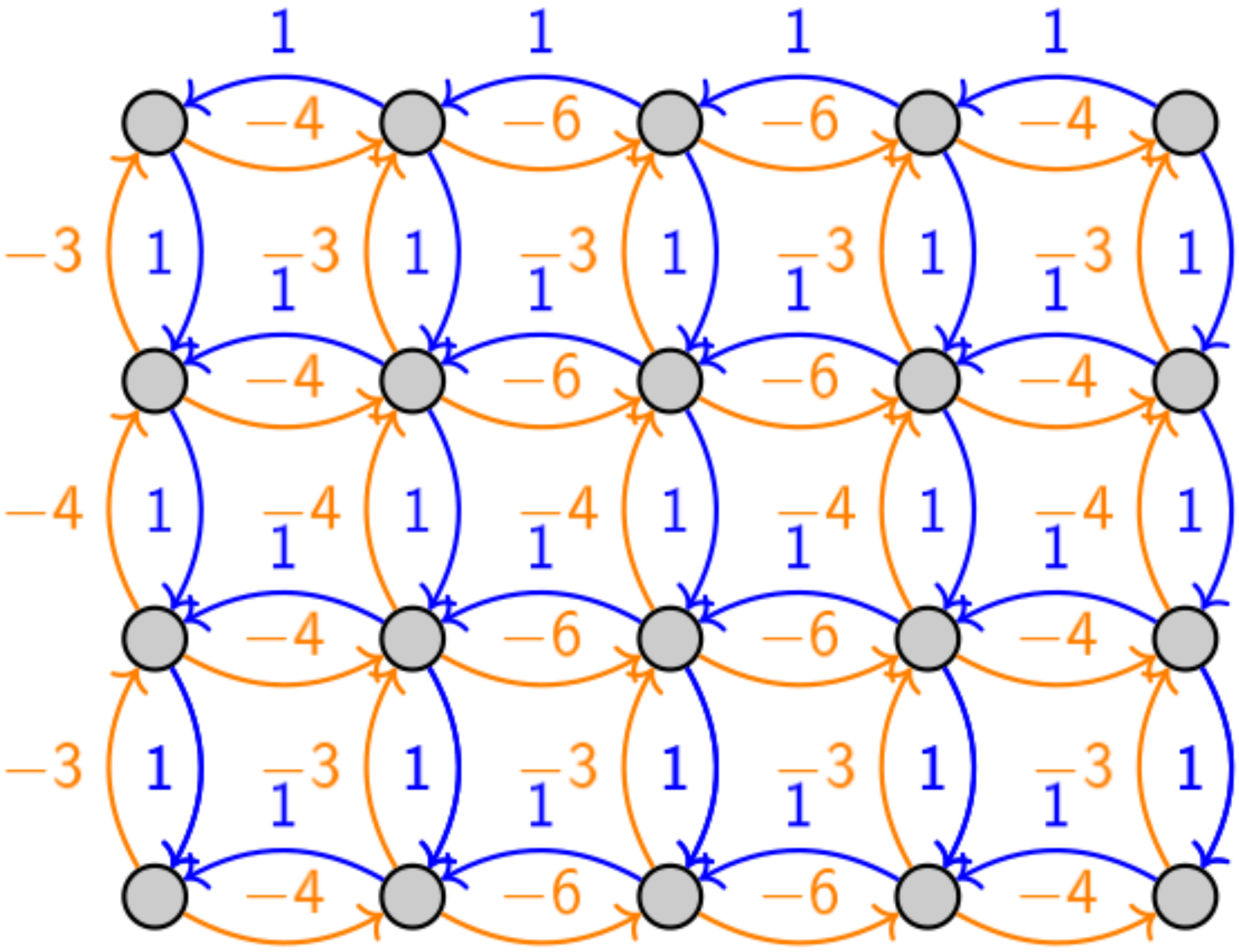
$$\rho(x)v_i = \alpha_i v_{i-1}, \quad \mu(x)v_i = \beta_i v_i,$$
$$\rho(y)v_i = \gamma_i v_i, \quad \mu(y)v_i = \delta_i v_{i+1},$$

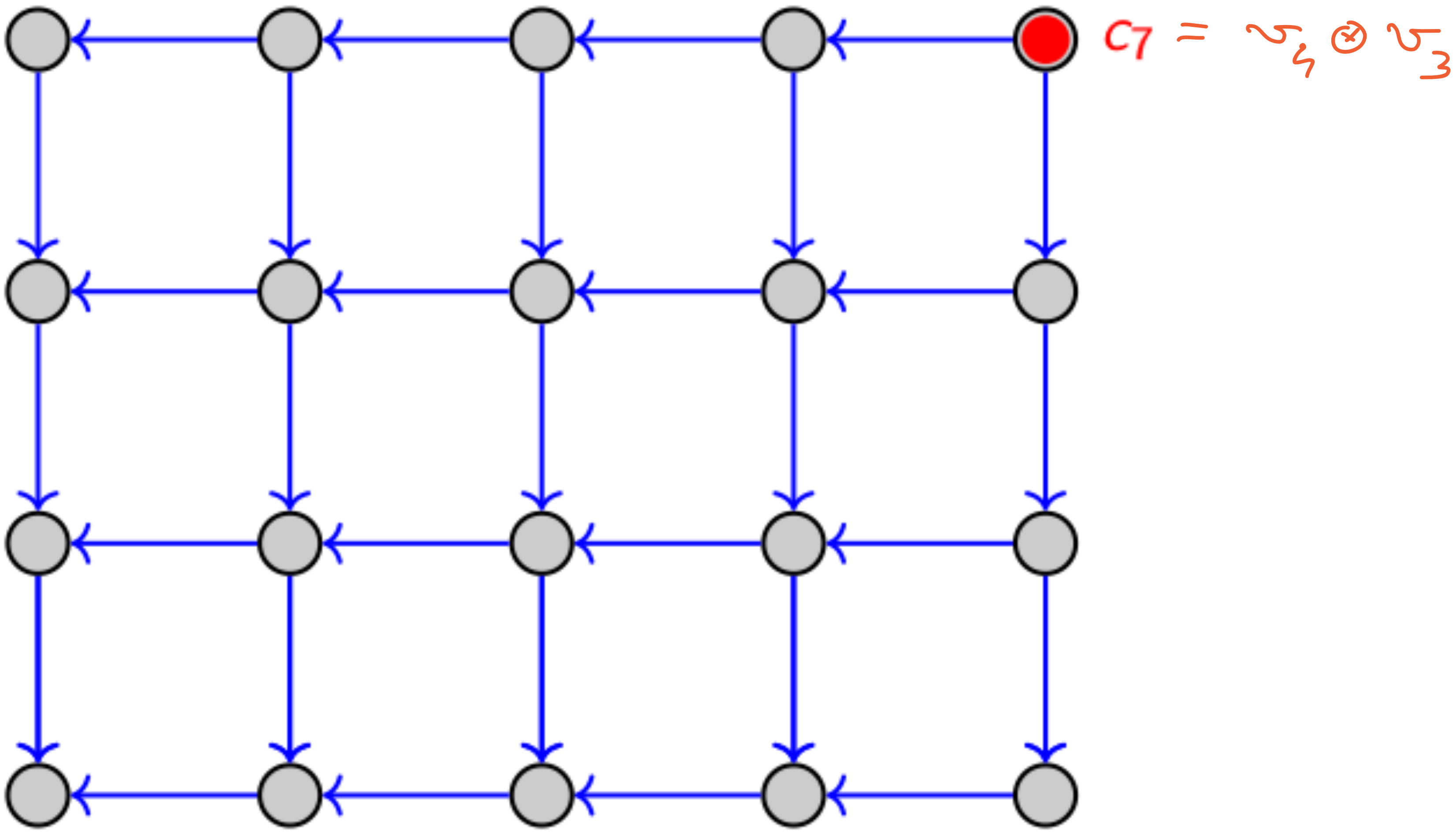
$V(4, \beta), \quad \beta \in \mathbb{F}$

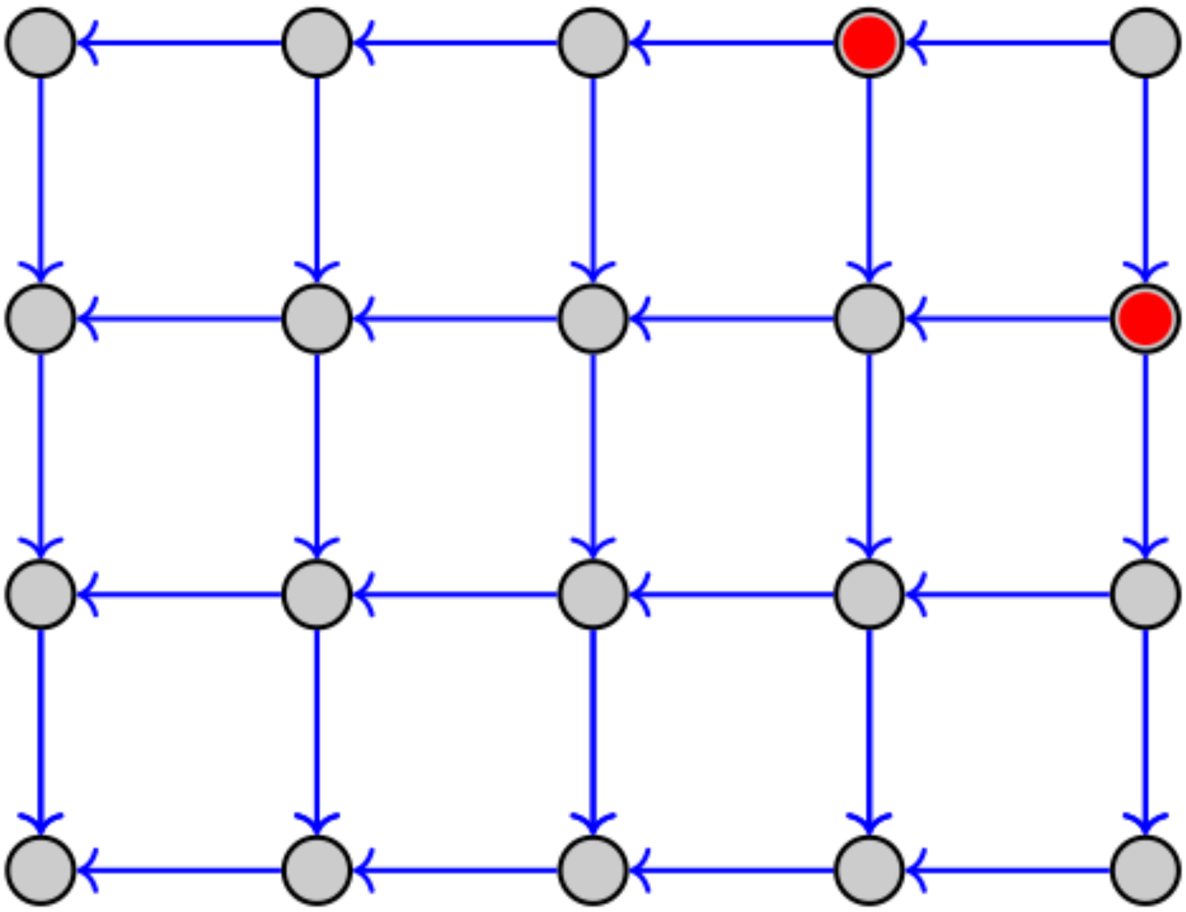
$\rho(x) \quad \rho(y) \quad \mu(x) \quad \mu(y)$



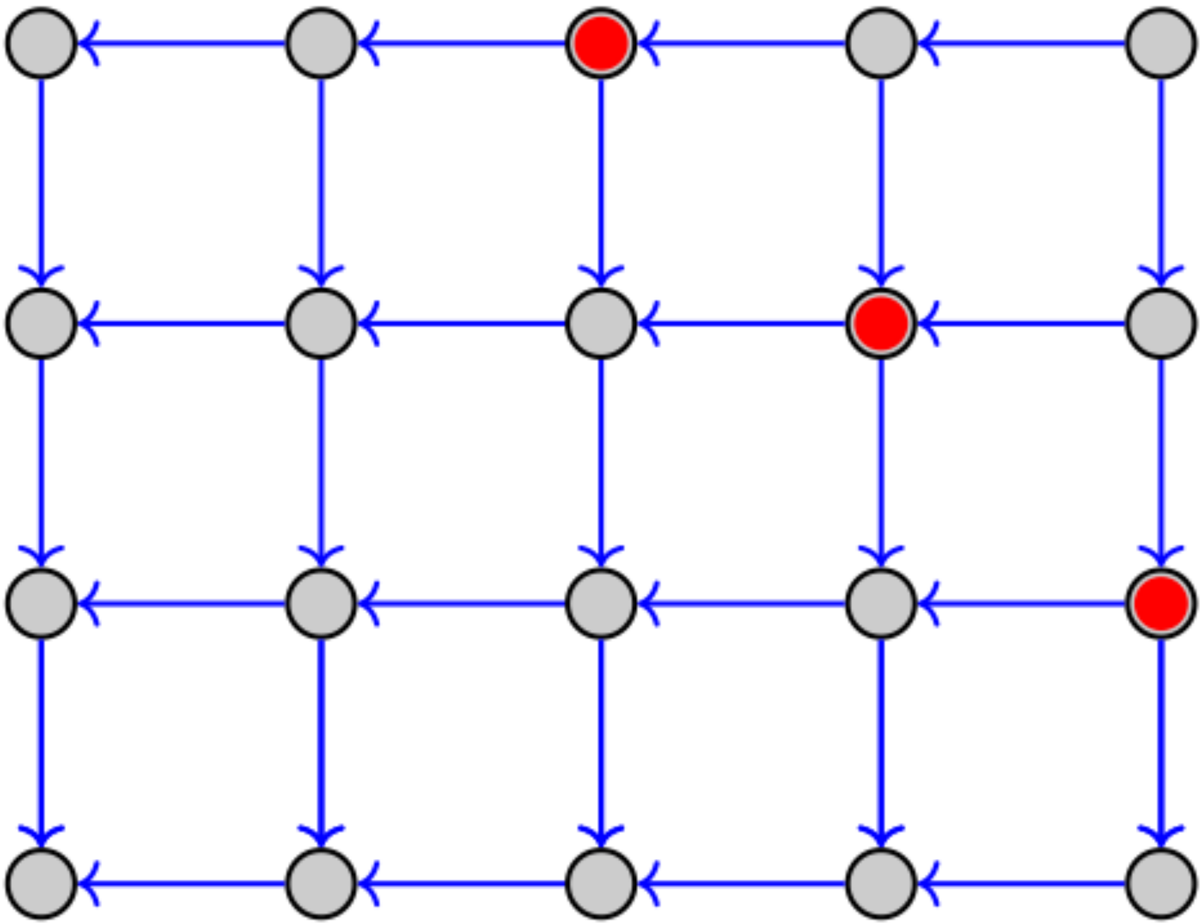
Example: $V(4,\beta) \otimes V(3,\beta')$




c7 = v4 ⊗ v3



c6 = $v_3 \otimes v_3 +$
$v_4 \otimes v_2$



$$c_5 = v_2 \otimes v_3 + 2 \cdot v_3 \otimes v_2 + v_4 \otimes v_1$$

Lemma

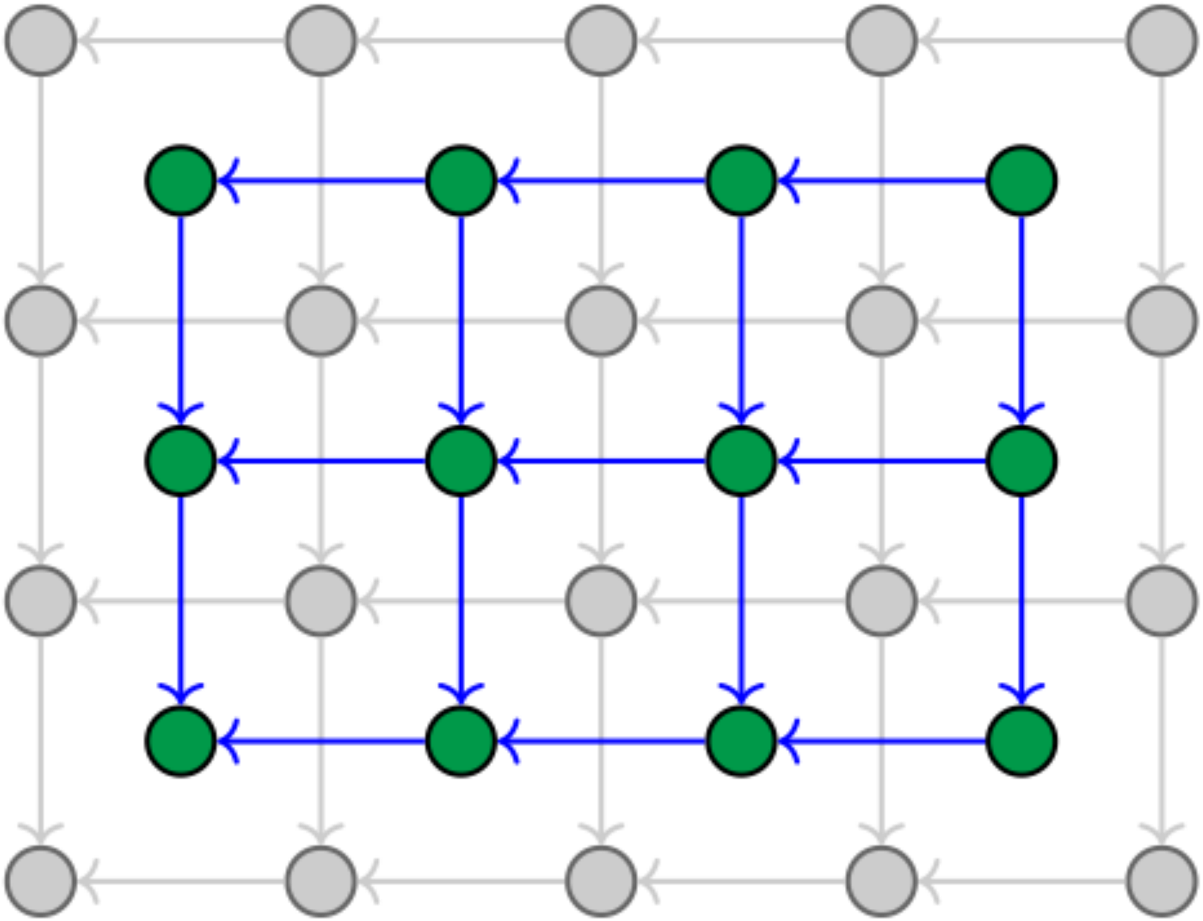
The vector subspace generated by $(\rho(x) \otimes \rho(x))^i(v_4 \otimes v_3)$ *is a subrepresentation of* $V(4, \beta) \otimes V(3, \beta')$ *isomorphic to* $V(7, \beta + \beta')$.

IN $V(m, \beta) \otimes V(n, \beta')$,

SPAN $\{(\rho(x) \otimes 1 + 1 \otimes \rho(x))^k (v_m \otimes v_n) : k \geq 0\}$

IS A SUBREP. ISOMORPHIC TO

$V(m+n, \beta+\beta')$



THIS «GREEN» REPRESENTATION IS ISOMORPHIC TO

$$V(3, \beta+1) \otimes V(2, \beta').$$

Theorem (Clebsch-Gordan formula)

We have that

$$V(m,\beta)\otimes V(n,\beta') \simeq \bigoplus_{i=0}^{\cancel{m}\ \mathrm{MIN}(m,n)} V(m+n-2i, \beta+\beta'+i).$$

$CL^{\alpha}_{2,4}$, $\alpha \in \mathbb{F}^2$

$$[x,y] = x$$

$$\{x,y\} = x + \alpha y$$

IS A COMPLETE LIST OF ISOMORPHISM CLASSES OF 2-DIM'L SIMPLE CLAs.