Kostant cuspidal permutations

Volodymyr Mazorchuk

(Uppsala University)

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Coauthor: Samuel Creedon

Setup

$\mathfrak{g}$ — semi-simple, finite dimensional Lie algebra over $\mathbb{C}$.

$U(\mathfrak{g})$ — the universal enveloping algebra of $\mathfrak{g}$.

M — $\mathfrak{g}$-module.

Then: $\mathrm{Hom}_{\mathbb{C}}(M, M)$ is a $\mathfrak{g}$-$\mathfrak{g}$-bimodule.

Adjoint action: $\mathrm{ad}_g(m) := g \cdot m - m \cdot g$.

Denote by $\mathcal{L}(M, M)$ the $\mathfrak{g}$-$\mathfrak{g}$-subbimodule of $\mathrm{Hom}_{\mathbb{C}}(M, M)$ consisting of all elements on which the adjoint action of $\mathfrak{g}$ is locally finite.

Note that the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ is locally finite.

Hence: $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \subset \mathcal{L}(M, M)$.

Kostant's problem

Kostant's problem: For which M, is the natural inclusion

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \subset \mathcal{L}(M, M)$$

an isomorphism?

Origin: Joseph, A. Kostant's problem, Goldie rank and the Gel'fand-Kirillov conjecture. Invent. Math. **56** (1980), no. 3, 191–213.

Status: Wide open, even for general simple highest weight modules.

Terminology: M is Kostant positive if the answer is "yes" and Kostant negative if the answer is "no".

Category $\mathcal{O}$

Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Consider the associated BGG category $\mathcal{O}$

and its principal block $\mathcal{O}_0$.

Let W be the Weyl group of $\mathfrak{g}$.

Then simples in $\mathcal{O}_0$ are in bijection with W.

Denote L_w the simple highest weight module with highest weight $w \cdot 0$ (the dot-action).

In particular, L_e is the trivial $\mathfrak{g}$-module and L_{w_0} is a simple Verma module.

Classical results

Theorem. (Joseph) The answer to Kostant's problem is positive for all Verma modules.

Observation. (Jantzen's book) The answer to Kostant's problem is positive for any quotient of any Verma module which is projective in $\mathcal{O}$.

Theorem. (Gabber-Joseph) Let $W^{\mathfrak{p}}$ be a parabolic subgroup of W. Let $w_0^{\mathfrak{p}}$ be the longest element of $W^{\mathfrak{p}}$. Then $L_{w_0^{\mathfrak{p}} w_0}$ is Kostant positive.

Time frame: Early 80's.

Importance

In late 90's, Milicic and Soergel developed some methods to establish equivalence between some categories of Lie algebra modules using Harish-Chandra bimodules.

These were further generalized and extended by Khomenko-M. and M.-Stroppel during the 00's.

Essential precondition: that Kostant's problem has the positive answer for a certain module.

More recent results

Theorem. (M., 2005) Let $W^{\mathfrak{p}}$ be a parabolic subgroup of W and $s \in W^{\mathfrak{p}}$ a simple reflection. Then $L_{sw_0^{\mathfrak{p}}w_0}$ is Kostant positive.

Theorem. (M.-Stroppel, 2008) In type A, the answer to Kostant's problem is an invariant of a Kazhdan–Lusztig left cells.

Theorem. (M.-Stroppel, 2008) In type A_3, namely $r \text{ —— } s \text{ —— } t$, the module L_{rt} is Kostant negative.

Theorem. (Kåhrström, 2010) Let $W^{\mathfrak{p}}$ be a parabolic subgroup of W and $w \in W^{\mathfrak{p}}$. Then the answers to Kostant's problem for $L^{\mathfrak{g}}_{ww_0^{\mathfrak{p}}w_0}$ and $L^{\mathfrak{g}^{\mathfrak{p}}}_{w}$ are the same (note that we are talking about different Lie algebras here).

More recent: A complete answer for L_w, where w is fully commutative (Mackaay-M.-Miemietz, 2024).

Full answer in small ranks, up to $\mathfrak{sl}_7$ (Kåhrström-M., 2010, Kåhrström, 2010, Ko-M.-Mrđen, 2020, Creedon-M., 2025).

Kåhrström's conjecture

Let θ_w be the indecomposable projective endofunctor of $\mathcal{O}_0$ corresponding to $w \in W$.

Conjecture (Kåhrström, 2019)

For an involution $d \in S_n$, the following assertions are equivalent:

(1) L_d is Kostant positive.

(2) if $x \neq y \in S_n$ are such that $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, then $\theta_x L_d \ncong \theta_y L_d$.

(3) if $x \neq y \in S_n$ are such that $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, then $[\theta_x L_d]^{\mathbb{Z}} \neq [\theta_y L_d]^{\mathbb{Z}}$ in the graded Grothendieck group of $\mathcal{O}_0$.

(4) if $x \neq y \in S_n$ are such that $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, then $[\theta_x L_d] \neq [\theta_y L_d]$ in the Grothendieck group of $\mathcal{O}_0$.

Note: for S_n it is enough to consider involutions.

Evidence: holds in all small rank cases.

Indecomposability conjecture

Conjecture. (Kildetoft-M., 2016)

Let $x, y \in S_n$ be such that $\theta_x L_y \neq 0$.

Then $\theta_x L_y$ is indecomposable.

Status: many special cases are known, but open in general.

Connection

Theorem. (Ko-M.-Mrđen, 2020)

Let $d \in S_n$ be an involution.

Then L_d is Kostant positive if and only if $\theta_x L_d$ is always indecomposable when non-zero

and, if $x \neq y \in S_n$ are such that $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, then $\theta_x L_d \ncong \theta_y L_d$.

Consecutive patterns

Let $p \in S_k$ and $w \in S_n$, for some $n \geq k$.

We say that w consecutively contains p starting from some position $i \leq n - k$ provided that

for any $s, t \in \{1, 2, \ldots, n\}$, we have $p(s) \leq p(t)$ if and only if $w(i - 1 + s) \leq w(i - 1 + t)$.

Example: the permutation 742815396 in S_9 (one-line notation)

consecutively contains 4132 (of S_4) starting from position 4

because we have 742815396 and the entries of 8153 are in the same relative order as the entries of 4132.

Original observation

Proposition. If some $w \in S_n$ consecutively contains 2143 (starting from some position), then w is Kostant negative.

Remark. $2143 \in S_4$ is exactly the smallest Kostant negative element in type A discovered by M.-Stroppel.

Corollary 1. We have

$$\lim_{n\to\infty} \frac{|\{w \in S_n \,:\, L_w \text{ is Kostant positive}\}|}{n!} = 0.$$

Corollary 2. We have

$$\lim_{n\to\infty} \frac{|\{w \in S_n \,:\, w^2 = e \text{ and } L_w \text{ is Kostant positive}\}|}{|\{w \in S_n \,:\, w^2 = e\}|} = 0.$$

Remark. The claims of both corollaries were conjectured in [MMM], where the corresponding claims were proved for fully commutative elements.

Main Theorem

Theorem. Let $p \in S_k$ be such that L_p is Kostant negative.

Let $w \in S_n$, for some $n \geq k$, be such that w consecutively contains p.

Then L_w is Kostant negative.

Remark. Note again that we connect here Kostant's problem for modules over different Lie algebras.

Kostant cuspidal permutations

Terminology: we say $w \in S_n$ is Kostant negative/ positive provided that L_w is.

Inspired by the main theorem we introduce:

Definition. An element $w \in S_n$ is called Kostant cuspidal

provide that it is Kostant negative

but any proper consecutive sub-pattern of w is Kostant positive.

Remark. Our terminology is inspired by O. Mathieu's terminology for weight modules.

Small rank examples

Example 1. $2143 \in S_4$ is Kostant cuspidal.

Example 2. There are five Kostant negative involutions in S_5:

$$21435, \quad 13254, \quad 21543, \quad 32154, \quad 14325.$$

Here we see that four of them contain 2143 as a consecutive pattern.

So, only 14325 is Kostant cuspidal.

Example 3. There are four Kostant cuspidal involutions in S_6 (out of 25 Kostant negative ones):

$$341265, \quad 215634, \quad 154326, \quad 426153.$$

Type A_6

In type A_6, there are 107 Kostant negative involutions.

Out of these exactly nine are Kostant cuspidal, namely:

$$1462537, \quad 1536247, \quad 3214765, \quad 4531276, \quad 3614725,$$

$$2167534, \quad 1654327, \quad 4271563, \quad 5237164.$$

The corresponding standard Young tableaux are:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3&7\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5&7\\\hline 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&7\\\hline 6\\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&7\\\hline 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&7\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5\\\cline{1-2} 4&7\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6\\\cline{1-2} 5&7\\\cline{1-2}\end{array}.$$

Left cell invariance

Theorem. Let $x, y \in S_n$ be two elements in the same left KL cell.

Then x is Kostant cuspidal if and only if y is.

Consequence: to answer Kostant's problem for simple highest weight modules in $\mathcal{O}_0$ in type A, we need to determine all Kostant cuspidal elements in all symmetric groups.

Each left KL cell in type A contains a unique involution.

Therefore it is enough to determine all Kostant cuspidal involutions.

Natural question: how many Kostant cuspidal involutions are there?

From small rank examples we see that the number of Kostant cuspidal involutions seems to be small but growing.

Kostant cuspidal families

Theorem.

(a) The only fully commutative Kostant cuspidal involutions of S_n are the elements

$$(i+1)(i+2)\dots(2i)12\dots i(j+1)(j+2)\dots n(2i+1)(2i+2)\dots j$$

(one-line notation), where $i = 1, 2, \dots, k-1$ and $j = 2i + \frac{n-2i}{2}$, for $n = 2k$ even.

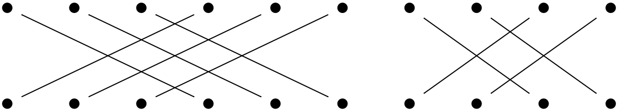
(b) For $n \geq 5$, the element $1(n-1)(n-2)\dots 2n$ (one-line notation) of S_n is Kostant cuspidal.

(c) For $n \geq 7$, the element $32145\dots(n-3)n(n-1)(n-2)$ (one-line notation) of S_n is Kostant cuspidal.

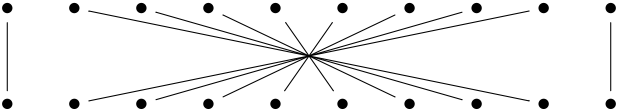
(d) For $n \geq 6$ and $i \in \{3, 4, \dots, n-3\}$, the product $(1, i+1)(i, n)$ of transpositions in S_n is Kostant cuspidal.

Explicit examples of cuspidal elements, I

Type (a)

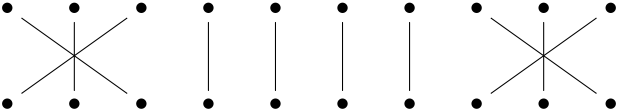


Type (b)

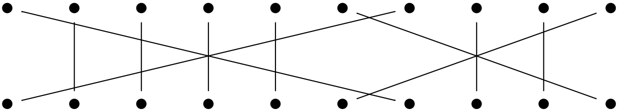


Explicit examples of cuspidal elements, II

Type (c)



Type (c)



Going back to small ranks

The only Kostant cuspidal involution 2143 in S_4 is type (a) (fully commutative).

The only Kostant cuspidal involution 14325 in S_5 is type (b).

Example 3. There four Kostant cuspidal involutions in S_6 are:

341265, 215634, 154326, 426153.

They are of type (a), type (b) and type (d).

Out of nine Kostant cuspidal involutions in S_7:

1462537, 1536247, 3214765, 4531276, 3614725,

2167534, 1654327, 4271563, 5237164.

These contain type (b), type (c) and type (d).

Consequences and comments

For $n \geq 6$, the number of Kostant cuspidal involutions in S_n is at least $n - 4$.

So, the growth of this number is bounded below at least by a polynomial of degree 1.

We do not expect the number of Kostant cuspidal involutions in S_n to be a monotone function (based on the behaviour of fully commutative elements depending on the parity).

To prove the Kostant cuspidality of some element one has to prove several things, including that it is Kostant negative (which is usually easier), but then also that the two maximal proper consecutive patterns in it are Kostant positive (which is usually much more difficult).

This provides some explanation why Kostant's problem is that difficult.

Comments one proofs

The proof of the main theorem is based on understanding Kåhrström's conjecture and its connection to Kostant's problem established by [KMM].

It also uses certain equivalence of categories that intertwine the action of projective functors discovered by Coulembier-M.-Zhang (in arXiv:1709.00547).

For the fully commutative cuspidal series, the proof is, essentially, a combinatorial argument relating what we need to the main result of [MMM].

The there remaining theorems are based on exploiting combinatorics of the Hecke algebra in relation to the combinatorial part of Kåhrström's conjecture as well as some GAP3 computations.

This includes, in some cases, explicit computation of the KL μ-function.

For this, in type (c), we, at some point, needed to use the AI inspired combinatorial description of the KL polynomials by G. Williamson and the DeepMind group (arXiv:2111.15161).

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THANK YOU!!!

Check out: Uppsala Algebra on YouTube:
https://www.youtube.com/channel/UCPWnhR29VHTAk7rZUEDQdDQ