Representations of generalized Weyl algebras

Representation Theory on Ice

Jonathan Nilsson

Linköping University

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in case of emergency
Luleå

Based on joint work with Samuel Lopes

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$\mathfrak{h}$ - a Cartan subalgebra

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Idea: What can be said about modules M where $\operatorname{Res}_{U(\mathfrak{h})}^{U(\mathfrak{g})} M$ is *free*?

Previous work

$$\mathfrak{C}_n = \{M \in U(\mathfrak{g})\text{-Mod} \mid \mathrm{Res}^{U(\mathfrak{g})}_{U(\mathfrak{h})} M \text{ is free of rank } n\}$$
$$\mathfrak{C}_{fg} = \{M \in U(\mathfrak{g})\text{-Mod} \mid \mathrm{Res}^{U(\mathfrak{g})}_{U(\mathfrak{h})} M \text{ is finitely generated}\}$$

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- Classification of $\mathfrak{C}_1$ for simple complex fin.dim. Lie algebras (N. 2015-2016)
  - Construction and classification for $\mathfrak{sl}_n$ and $\mathfrak{sp}_n$
  - Completion of classification via Weighting functors (idea by O. Mathieu)
- Simple $\mathfrak{sl}_2$-modules in $\mathfrak{C}_n$ for arbitrary rank n (F. Martin, C. Prieto 2017)
- Tensor modules - $\mathfrak{sl}_n$ modules of arbitrary rank (D. Grantcharov, K. Nguyen 2020)
- Generalizations of $\mathfrak{C}_n$ for related algebras:
  - Virasoro algebras (G. Liu, K. Zhao), Conformal algebras (Q. Xie et al.), The Witt algebra (H. Tan, K. Zhao), Algebras of differential operators (S. Gao et al.), Heisenberg-Virasoro algebras (H. Chen, X. Guo), Super Lie algebras (Y. Cai, K. Zhao), Kac-Moody algebras (K. Zhao et al.), Smith algebras (V. Futorny, S. Lopes, E. Mendonça)
- $U(\mathfrak{h})$-finite modules and weighting functors (E. Mendonça 2025)
- Classification of scalar type $\mathfrak{sl}_2$-modules in $\mathfrak{C}_2$ (D. Grantcharov, K. Nguyen, K. Zhao 2026)

Weighting functor

$\lambda : \mathfrak{h} \to \mathbb{C}$ lifts to $\overline{\lambda} : U(\mathfrak{h}) \to \mathbb{C}$.

$$\mathcal{W} : U(\mathfrak{g})\text{-Mod} \to U(\mathfrak{g})\text{-Mod} \qquad M \mapsto \bigoplus_{\lambda \in \mathbb{C}} M/\ker(\overline{\lambda})M$$

with natural $\mathfrak{h}$-action and where root vectors act by

$$x_\alpha \cdot (m + \ker(\overline{\lambda})) := (x_\alpha \cdot m) + \ker(\overline{\lambda + \alpha})$$

Properties of the weighting functor

- $\mathcal{W}(M)$ is a weight module
- M is a weight module $\Leftrightarrow \mathcal{W}(M) \simeq M$
- $\mathcal{W} \circ \mathcal{W} \simeq \mathcal{W}$
- $M \in \mathfrak{C}_n \quad \implies \quad \mathcal{W}(M)$ is a *coherent family* of degree $n \quad \implies \quad \mathfrak{g}$ is of type A or C.

$\mathfrak{sl}_2$-modules free of rank 1 setting

$\mathfrak{sl}_2 = \mathrm{span}(\mathrm{x}, \mathrm{h}, \mathrm{y})$. Let $M = U(\mathfrak{h}) = \mathbb{C}[h]$. Suppose there is an $\mathfrak{sl}_2$ action on M.
Set $p(h) := x \cdot 1$ and $q(h) := y \cdot 1$.
Then $x \cdot f(h) = f(h-2)p(h)$ and $y \cdot f(h) = f(h+2)q(h)$.

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$$x \cdot y \cdot f(h) - y \cdot x \cdot f(h) = h \cdot f(h) \Leftrightarrow p(h)q(h-2) - q(h)p(h+2) = h$$

$$\Leftrightarrow g(h) - \sigma^{-1}(g(h)) = h \text{ where } g(h) = p(h)q(h-2), \sigma(f(h)) = f(h-2)$$

$$\Leftrightarrow g(h) = -\frac{1}{4}(h^2 - 2h + c) = \frac{1}{4}(h+b)(h-b-2) \quad b \in \mathbb{C}$$

$$\Leftrightarrow p(h)q(h-2) = -\frac{1}{4}(h+b)(h-b-2)$$

Three types $M(p(h))$

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For $b \in \mathbb{N}$: $0 \to M(h-b-2) \to M(h+b) \to L(b) \to 0$

Part I

Generalized Weyl Algebras

Definition (V. Bavula 1992)

Let R be a ring, $a \in Z(R)$ a central element, and $\sigma \in \mathrm{Aut}(R)$ a ring automorphism. The corresponding **Generalized Weyl Algebra** $A = R(\sigma, a)$ is the ring generated by R and two variables x and y with relations

$$x\sigma(r) = rx \qquad yr = \sigma(r)y \qquad xy = a \qquad yx = \sigma(a)$$

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Variations:

- Higher rank GWAs (Bavula)
- Weak GWAs (Mazorchuk-Turowska)
- Twisted GWAs (Lu, Mazorchuk, Zhao)

Properties of the GWA $A = R(\sigma, a)$

- A is $\mathbb{Z}$-graded
- A is a free R-module with basis $\{1, x^n, y^n \mid n \in \mathbb{Z}_{>0}\}$
- A is a (noetherian) domain iff R is a (noetherian) domain and $a \neq 0$.
- A is simple if and only if
  - a is not a zero divisor
  - R has no nontrivial σ-stable ideals
  - $R = Ra + R\sigma^n(a)$ for all $n > 0$
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For the rest of this talk consider a GWA $A = R(\sigma, a)$ where R be a UFD and $a \neq 0$.

Realizing various algebras as GWAs

Algebra $R(\sigma, a)$	**Ring** R	**Automorphism** σ	**Element** a
Classical Weyl Algebra	$k[h]$	$\sigma(h) = h - 1$	h
Quantum torus	$k[t, t^{-1}]$	$\sigma(t) = qt$	t
Quantum Weyl Algebra	$k[h]$	$\sigma(h) = qh - 1$	h
$U(\mathfrak{sl}_2)$	$\mathbb{C}[h, c]$	$\sigma(h) = h - 1$, $\sigma(c) = c$	$c - h(h+1)$
$U(\mathfrak{sl}_2)$/central action	$\mathbb{C}[h]$	$\sigma(h) = h - 1$	$c - h(h+1)$
Quantum $\mathfrak{sl}_2$	$k[s^{\pm 1}, t]$	$\sigma(t) = t + \frac{s - s^{-1}}{q - q^{-1}}, \sigma(s) = q^{-2}s$	t
Smith algebras/central action	$k[h]$	$\sigma(h) = h - 1$	$p(h) \in k[h]$
Gen. Down-Up algebras	$k[h, t]$	$\sigma(h) = f(h), \sigma(t) = qt + g(h)$	t
$\mathrm{GL}_q(2)$	$k[b, c, \Delta^{\pm 1}]$	$\sigma : b \mapsto q^{-1}b, c \mapsto q^{-1}c, \Delta \mapsto \Delta$	$\Delta + qbc$

Weight modules for GWAs

$V \in R(\sigma, a)$-Mod. For each maximal ideal $\mathfrak{m} \lhd R$ we define the corresponding **weight space**:

$$V_{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}v = 0\}.$$

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Verma-like construction

Suppose $\mathfrak{m} \in \operatorname{Max}(R)$ with infinite σ-orbit: $|\sigma^{\mathbb{Z}}(\mathfrak{m})| = \infty$, and suppose that $a \in \mathfrak{m}$. Let $R/\mathfrak{m}$ be the $R\langle y\rangle$-module with natural R-action and trivial y-action. Set

$$M(\mathfrak{m}) = R(\sigma, a) \otimes_{R\langle y\rangle} R/\mathfrak{m}.$$

Then $M(\mathfrak{m})$ has a unique maximal sub and corresponding simple quotient simple quotient $L(\mathfrak{m})$.

Visualization

Visualization of a simple weight module $V = L(\mathfrak{m})$ with finite support:

$$\begin{tikzcd}
0 & \boxed{V_{\mathfrak{m}}} \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] \arrow[loop above, "R"] & \boxed{V_{\sigma^{-1}(\mathfrak{m})}} \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] \arrow[loop above, "R"] & \boxed{V_{\sigma^{-2}(\mathfrak{m})}} \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] \arrow[loop above, "R"] & \boxed{V_{\sigma^{-3}(\mathfrak{m})}} \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] \arrow[loop above, "R"] & \boxed{V_{\sigma^{-4}(\mathfrak{m})}} \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] \arrow[loop above, "R"] & 0
\end{tikzcd}$$

Part II

R-free modules

Two category of modules for GWAs

Definition

For a GWA $R(\sigma, a)$, consider the full subcategories of $R(\sigma, a)$-Mod:

$$\mathfrak{C}_{fg} = \{M \in R(\sigma, a)\text{-Mod} \mid M \text{ is finitely generated over } R\}.$$

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This is an abelian category. We also define also $\mathfrak{C}_n \subset \mathfrak{C}_{fg}$:

$$\mathfrak{C}_n = \{M \in R(\sigma, a)\text{-Mod} \mid M \underset{R}{\simeq} R^n\},$$

Objects of $\mathfrak{C}_1$

Definition

Let $R(\sigma, a)$ be a GWA over a UFD R. Let $p|a$ be divisor, and define $q = \sigma(a/p)$. We define a corresponding $R(\sigma, a)$-module V_p, which as a set (and R-module) is R and where the action of x and y is given by:

$$x \cdot r = \sigma^{-1}(r)p \quad \text{and} \quad y \cdot r = \sigma(r)q.$$

Properties of $\mathfrak{C}_1$

- Any module in $\mathfrak{C}_1$ is isomorphic to some V_p.
- For $u \in R^\times$, $V_p \simeq V_{u^{-1}p\sigma^{-1}(u)}$
- If σ fixes units $V_p \simeq V_{p'} \Leftrightarrow p = p'$
- V_p and V_{up} are related via twisting by automorphisms: $V_{up} \simeq V_p^{\tau_u}$, where $\tau_u \in Aut(A)$ sends $r \mapsto r$, $x \mapsto ux$, $y \mapsto yu^{-1}$.
- For $R = \mathbb{F}[t]$, the nonproper divisor $p = \xi a$ for $\xi \in \mathbb{F}$ produces a class of *Whittaker-modules* which were previously studied by Benkart and Ondrus in 2008.

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So if $g = \prod g_i$ is a complete factorization, we get

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$$g_i|\sigma^{-1}(g) \text{ or } g_i|p \qquad \text{and} \qquad g_i|\sigma(g) \text{ or } g_i|q.$$

So if g is a *chain product* $g = \prod_{i=0}^{n} \sigma^i(z)$, this gives $z|q$ and $\sigma^n(z)|p$

σ-orbits

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Then $\langle g\rangle \subset V_p$ is a $R(\sigma, a)$-submodule if and only if $\langle g_\omega\rangle \subset V_{p_\omega}$ is a $R(\sigma, a_\omega)$-submodule for each $\omega \in \Omega$.

Submodules - infinite orbit case

Assume that all factors of a lie in a single *infinite* σ-orbit.
Then the maximal submodules of V_p have form $W = \langle g \rangle$ where g is a chain-product

$$g = \prod_{i=0}^{n} \sigma^i(z)$$

where $z|q$ and $\sigma^n(z)|p$, and such that the factors in the middle of the chain divides neither p nor q.

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We note that x acts on the R-generator g of $\langle g \rangle$ as

$$x \cdot g = p\sigma^{-1}(g) = p \cdot \prod_{i=-1}^{n-1} \sigma^i(z) = \frac{p}{\sigma^n(z)} \sigma^{-1}(z) g$$

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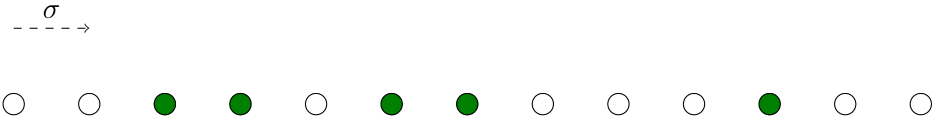
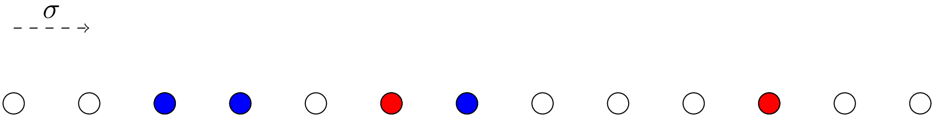
so $\langle g \rangle \simeq V_{p'}$ where $p' = \frac{p}{\sigma^n(z)} \sigma^{-1}(z)$.



factors of a



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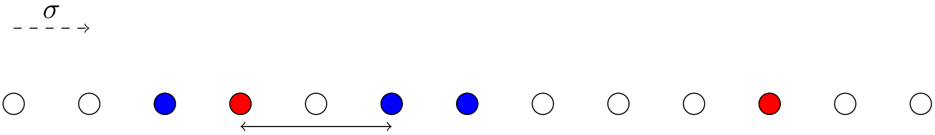
factors of $\sigma^{-1}(q)$



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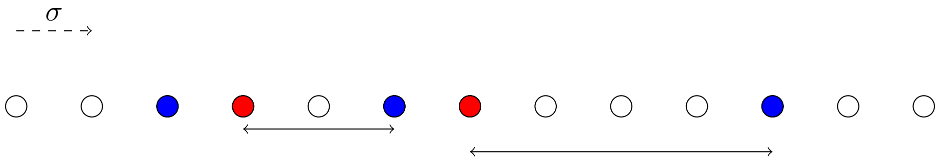

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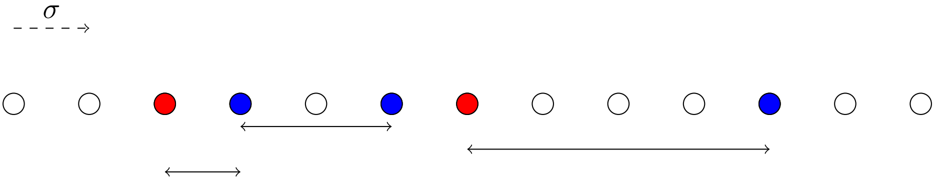

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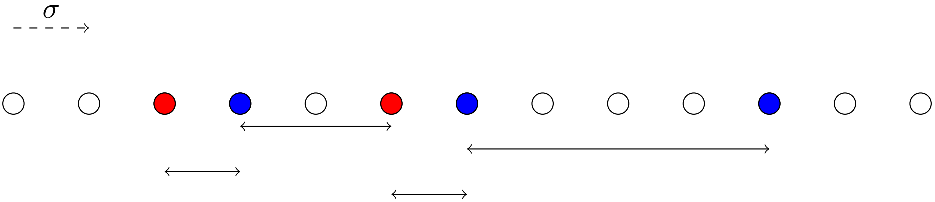

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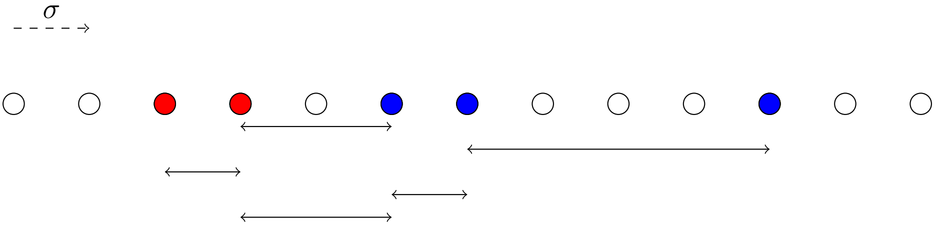

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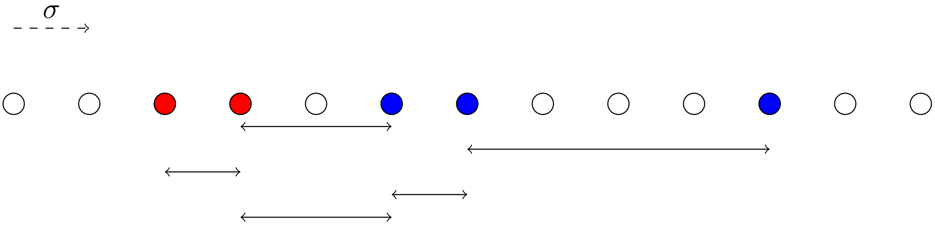

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Quotients

Assume that σ-orbits are infinite. Let $\langle g \rangle$ be a maximal submodule of V_p, with $g = \prod_{i=0}^{n} \sigma^i(z)$ and $z|q$ and $\sigma^n(z)|p$.
As R-modules we have

$$V_p/\langle g \rangle \simeq R/\langle z \rangle \oplus R/\langle \sigma(z) \rangle \oplus \cdots \oplus R/\langle \sigma^n(z) \rangle,$$

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As R-modules we have

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This is in fact isomorphic to the simple *weight module* $L(\mathfrak{m})$ for $\mathfrak{m} = \langle \sigma^n(z) \rangle$.

Length in infinite orbits case

The length of the module V_p is $1+$the *number of flips* in our diagram. When a is square free this is equal to the number of pairs (p_i, q_i) of irreducible factors of p and q respectively where $p_i \in \sigma^{\mathbb{N}}(q_i)$.

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The length of the module V_p is $1+$the *number of flips* in our diagram. When a is square free this is equal to the number of pairs (p_i, q_i) of irreducible factors of p and q respectively where $p_i \in \sigma^{\mathbb{N}}(q_i)$.

We get a bound

$$\operatorname{len}(V_p) \leq \left(\frac{\deg(a)}{2}\right)^2 + 1$$

and $\mathfrak{C}_1$ a finite length category.

Grothendieck group

Grothendieck group $K_0(\mathfrak{C})$: abelian group generated by iso-classes of modules in $\mathfrak{C}$ with relations

$$[A] + [C] = [B] \quad \text{for each short exact sequence} \quad 0 \to A \to B \to C \to 0.$$

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Theorem

Assume that all σ-orbits are infinite. Then in $K_0(\mathfrak{C}_{fg})$ we have

$$[V_p] = [V_{\hat{p}}] + \sum_{a \in \mathfrak{m}} n_{\mathfrak{m}} L(\mathfrak{m})$$

where $V_{\hat{p}} = \mathrm{soc}(V_p)$ is simple, the occurring $L(\mathfrak{m})$ have finite support, and the coefficients $n_{\mathfrak{m}} \in \mathbb{N}_0$ can be expressed combinatorially.

Example over $\mathfrak{sl}_2$

For a fixed $b \in \mathbb{C}$, let $R = \mathbb{C}[h]$, $\sigma(f(h)) = f(h-2)$, and $a = -\frac{1}{4}(h+b)(h-b+2)$.

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$$\sigma^k(h+b) = h-b+2 \Leftrightarrow h+b-2k = h-b+2 \Leftrightarrow k = b-1$$

So all four modules are simple unless $b \in \mathbb{Z}$, and for $b \in \mathbb{N}$ we have a composition series $\{0\} \subset V_{(h+b-2)} \subset V_{(h-b)}$ where the top is the b-dimensional simple weight module.

Finite orbit case

Suppose that there is $z \in \mathrm{Irr}(R)$ such that $\sigma^m(z) = z$.

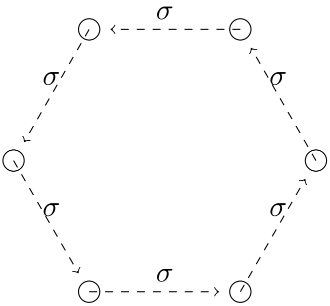
Take $g = \prod_{i=0}^{m-1} \sigma^i(z)$. Then $\sigma(g) = g$. So $\langle g \rangle$ a submodule.

We have an infinite chain of submodules

$$\cdots \subset \langle g^3 \rangle \subset \langle g^2 \rangle \subset \langle g \rangle \subset V_p$$

So no V_p has finite length.

Finite orbit example



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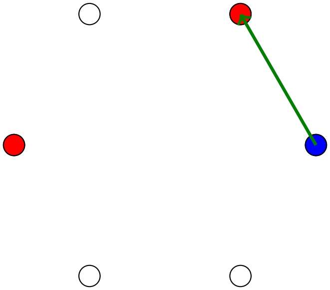

factors of p

factors of $\sigma^{-1}(q)$

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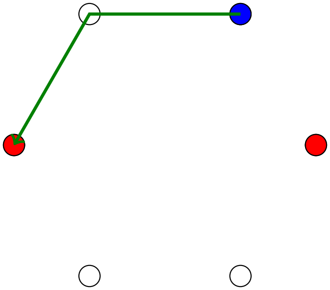

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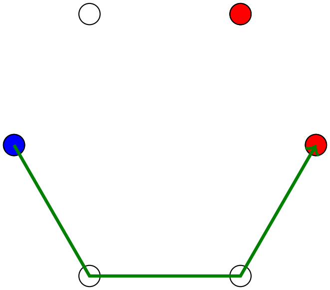

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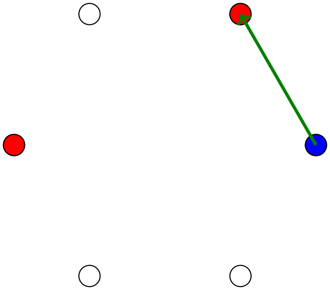

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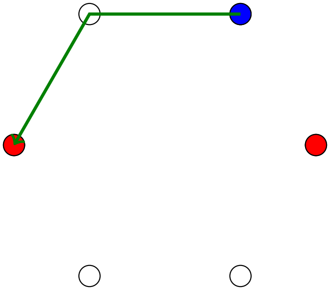

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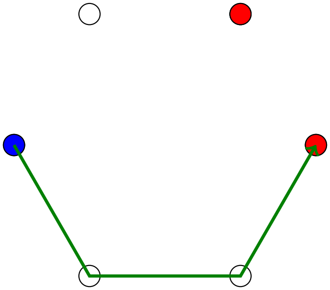

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So in this case we get an infinite 3-periodic "composition series"

$$\cdots \subset V_p \subset V_{p''} \subset V_{p'} \subset V_p \subset V_{p''} \subset V_{p'} \subset V_p$$

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In the general case we get a finite set of simple weight modules appearing as subquotients of V_p.

A weighting functor for GWA-modules

Let $A = R(\sigma, a)$ where R is a UFD. Define

$$\mathcal{W} : A\text{-Mod} \to A\text{-Mod}$$

where $\mathcal{W}(M) = \bigoplus_{\mathfrak{m} \in \mathrm{Max}(R)} M/\mathfrak{m}M$ with natural R-action, and

$$x \cdot (m + \mathfrak{m}M) := (x \cdot m) + \sigma^{-1}(\mathfrak{m})M \qquad y \cdot (m + \mathfrak{m}M) := (y \cdot m) + \sigma(\mathfrak{m})M$$

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- $\mathcal{W}(M)$ is a weight module
- M is a weight module $\Leftrightarrow \mathcal{W}(M) \simeq M$
- $\mathcal{W} \circ \mathcal{W} \simeq \mathcal{W}$
- $M \in \mathfrak{C}_n \implies \mathcal{W}(M)$ is a weight-module analogous to a coherent family.

Part III

Higher rank

Higher rank

Let $A = R(\sigma, a)$ be a GWA over a PID R, and pick $\mathsf{P} \in \mathrm{Mat}_n(R)$ with $\det(\mathsf{P})|a$. Define $\mathsf{Q} = \sigma(a\mathsf{P}^{-1}) \in \mathrm{Mat}_n(R)$. Then $V_{\mathsf{P}} = R^n$ becomes an A-module under the actions

$$x \cdot v = \mathsf{P}\sigma^{-1}(v) \quad y \cdot v = \mathsf{Q}\sigma(v)$$

where $v = (r_1, \ldots, r_n)^T$ and automorphisms apply coordinate-wise to v.

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- Any module in $M \in \mathfrak{C}_n$ is isomorphic to some V_P.
- $V_{\mathrm{P}} \simeq V_{\mathrm{P}'}$ iff $\mathrm{P}' = S^{-1}P\sigma^{-1}(S)$ for some $S \in \mathrm{GL}_n(R)$

Stratification by Smith Normal Form

Given a module V_{P} we can express

$$\mathrm{P} = SDT$$

with $S, T \in \mathrm{GL}_n(R)$, $D = diag(d_1, \ldots, d_n)$ where $d_i | d_{i+1}$. Write $\mathrm{SNF}(\mathrm{P}) = (d_1, \ldots, d_n)$.

- SNF is an isomorphism invariant
- $\mathrm{SNF} = (1, \cdots, 1)$ in Martin-Prieto's paper
- For $R(\sigma, a) \simeq U(\mathfrak{sl}_2)$ and $\mathrm{SNF}(\mathrm{P}) = (1, 1)$, classification of V_{P} in Grantcharov-Nguyen-Zhao's paper

Another class of simple rank n modules for GWAs

Theorem

Let $A = R(\sigma, a)$ be a GWA over a PID R. Suppose all σ-orbits are infinite. Let a_0 be an irreducible divisor of a. Let $V_n(a_0) = R^n$. Define an action of $R(\sigma, a)$ on $V_n(a_0)$ by

$$x \cdot re_i = \sigma^{-1}(r)e_{i+1} \text{ for } i < n, \text{ and } x \cdot re_n = a_0\sigma^{-1}(r)e_1$$

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Then $V_n(a_0)$ is a **simple** $R(\sigma, a)$-module which is free of rank n over R.
$V_n(a_0) = V_{\mathrm{P}}$ with $\mathrm{SNF}(\mathrm{P}) = (1, \ldots, 1, a_0)$.

Matrix version

$V_n(a_0) = V_{\mathrm{P}}$ with P and $\mathrm{Q} = \sigma(a\mathrm{P}^{-1})$ given by

$$\mathrm{P} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad \mathrm{Q} = \begin{pmatrix} 0 & \sigma(a) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma(a) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sigma(a) & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sigma(a) \\ \sigma\left(\frac{a}{a_0}\right) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

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Note that $\mathrm{P} = \mathrm{Comp}(t^n - a_0)$, the companion matrix of $p(t) = t^n - a_0 \in R[t]$.

x^n acts diagonally on V_{P}:

$$x^n \cdot v = \begin{pmatrix} a_0 & & & \\ & \sigma^{-1}(a_0) & & \\ & & \ddots & \\ & & & \sigma^{-(n-1)}(a_0) \end{pmatrix} \sigma^{-n}(v) = \begin{pmatrix} a_0\sigma^{-n}(v_1) \\ \sigma^{-1}(a_0)\sigma^{-n}(v_2) \\ \vdots \\ \sigma^{-(n-1)}(a_0)\sigma^{-n}(v_n) \end{pmatrix}.$$

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Suppose $w_k \neq 0$. Then $W \ni w' = w_k \cdot (x^n \cdot w) - \sigma^{-n}(w_k)\sigma^{-(k-1)}(a_0) \cdot w$ has one more zero than w.

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$\cdots \Rightarrow re_i \in W$.

An embedding

We have

$$A = R(\sigma, a) \supset \langle 1, x^n, y^n \rangle \simeq R(\sigma^n, a\sigma(a)\cdots\sigma^{n-1}(a)) = A'$$

Then Re_i is a R-free module of rank 1 over the GWA A'. By simplicity-results in the rank 1 case, Re_i is simple, so $e_i \in W$ for all i and $W = R^n$.

Let $\mathfrak{sl}_2$ be Lie algebra with basis $\{x, y, h\}$ over a field k of characteristic 0, and with relations $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$. For $b \in k \backslash \mathbb{Z}$ and $n \in \mathbb{N}$, let $V_n^{(b)} = k[t]^n$ and define an action of $\mathfrak{sl}_2$ on $V_n^{(b)}$ by

$$h \cdot \begin{bmatrix} f_1(h) \\ f_2(h) \\ \vdots \\ \vdots \\ \vdots \\ f_n(h) \end{bmatrix} = \begin{bmatrix} hf_1(h) \\ hf_2(h) \\ \vdots \\ \vdots \\ \vdots \\ hf_n(h) \end{bmatrix} \qquad x \cdot \begin{bmatrix} f_1(h) \\ f_2(h) \\ \vdots \\ \vdots \\ \vdots \\ f_n(h) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & h-b \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} f_1(h-2) \\ f_2(h-2) \\ \vdots \\ \vdots \\ \vdots \\ f_n(h-2) \end{bmatrix}$$

$$y \cdot \begin{bmatrix} f_1(h) \\ f_2(h) \\ \vdots \\ \vdots \\ \vdots \\ f_n(h) \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 0 & \theta & 0 & 0 & \cdots & 0 \\ 0 & 0 & \theta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \theta & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \theta \\ t+b & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(h+2) \\ f_2(h+2) \\ \vdots \\ \vdots \\ \vdots \\ f_n(h+2) \end{bmatrix}$$

where $\theta := (h-b+2)(h+b) = h^2 + 2h - b(b-2)$.

Under this action, $V_n^{(b)}$ is a simple $\mathfrak{sl}_2$-module which is free of rank n over the subalgebra $U(\mathfrak{h})$.

For $b \in k \setminus \mathbb{Z}$, let $V_2^{(b)} = k[h] \oplus k[h]$ and define an action of $\mathfrak{sl}_2$ on $V_2^{(b)}$ by

$$h \cdot \begin{bmatrix} f_1(h) \\ f_2(h) \end{bmatrix} = \begin{bmatrix} hf_1(h) \\ hf_2(h) \end{bmatrix} \qquad x \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 0 & h-b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1(h-2) \\ f_2(h-2) \end{bmatrix}$$

$$y \cdot \begin{bmatrix} f_1(h) \\ f_2(h) \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 0 & (h-b+2)(h+b) \\ h+b & 0 \end{bmatrix} \begin{bmatrix} f_1(h+2) \\ f_2(h+2) \end{bmatrix}$$

Under this action, $V_2^{(b)}$ is a **simple** $\mathfrak{sl}_2$-module which is free of rank 2 over the subalgebra $U(\mathfrak{h})$.

Open questions

- Simplicity of V_p when R is a UFD
- Classification/simplicity for modules of rank n over R
- Rank n GWAs
- Twisted GWAs
- Weak GWAs
- Other ring-extension: Ore extensions, ambiskew polynomial rings, skew group rings, crossed product algebras, smash products
- Weighting functors on $\mathfrak{C}_{fg}$

Thanks!