Radicals, quivers & species for algebra objects

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- Vaguely vertex-friendly motivation:

. $\mathcal{C}$ = "vertex $\otimes$-category" [Huang-Lepowsky]

. $\mathcal{M}$ = boundary conditions in Fuchs-Runkel-Schweigert construction on $\mathcal{C}$

Some representation-theoretic examples

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$\mathrm{Rep}(\mathfrak{g}) \curvearrowright \mathcal{O}$ **(3)**

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$$\mathrm{Rep}(\mathfrak{g}) \curvearrowright \mathcal{O}$$

$$\mathcal{O} = \bigoplus_{\chi} \mathcal{O}_\chi \rightsquigarrow \mathcal{P}(\mathfrak{g}) = \bigoplus_{\chi,\chi'} \mathcal{P}_{\chi,\chi'}$$

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{g}) \arrow[r, "P"] \arrow[dr, two heads] & \mathrm{End}(\mathcal{O}) \\ & \mathcal{P}(\mathfrak{g}) \arrow[u, hook] \end{tikzcd}$$

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4

$$\begin{tikzcd} \mathcal{P}_{0,0} \arrow[r, bend left] \arrow[d] & \mathcal{O}_0 \\ \overline{\mathcal{P}}_{0,0} \arrow[ur, bend left] & \end{tikzcd}$$

shuffling

Different kinds of monoidal categories

	semisimple	finite	rigid (duals)
① $\mathrm{Rep}_{\mathbb{C}}(G)$	✓	✓	✓
② $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(\mathbb{Z}/p\mathbb{Z})$	×	✓	✓
③ $\mathrm{Rep}(\mathfrak{g})$	✓	×	✓
④ $\overline{\mathcal{P}}_{0,0}$	×	✓	almost

Different kinds of monoidal categories

some ⊗-words	semisimple	finite	rigid (duals)
fusion cat	✓	✓	✓
finite ⊗-cat	×	✓	✓
	✓	×	✓
2-rep. [Mazorchuk-Miemietz]	×	✓	almost

Different kinds of monoidal categories

some vertexy words	some $\otimes$-words	semisimple	finite	rigid (duals)
regular	fusion cat	✓	✓	✓
rational C_2-cofinite	finite $\otimes$-cat	×	✓	✓
		✓	×	✓
triplets $W_{p,q}$	2-rep. [Mazorchuk - Miemietz]	×	✓	almost

Different kinds of monoidal categories

TODAY'S RESULTS	some $\otimes$-words	semisimple	finite	rigid (duals)
100%	fusion cat	✓	✓	✓
75%	finite $\otimes$-cat	×	✓	✓
Work in progress		✓	×	✓
	2-rep. [Mazorchuk-Miemietz]	×	✓	almost

Module categories are categories of modules

Thm [Etingof-Ostrik]. For $\mathcal{C}$ a finite $\otimes$-category

$$\left\{ \mathcal{C}\text{-modules} \right\}_{/\simeq} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{Algebra objects} \\ \text{in } \mathcal{C} \end{array} \right\}_{/\text{Morita}}$$

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$$\begin{array}{c}\mathrm{mod}_{\mathcal{C}}(A) \\ \shortparallel \\ \left\{(M,\ M\otimes A \xrightarrow[h]{} M)\right\}\end{array} \longleftarrow (A,\ \mu: A\otimes A \to A,\ \eta: \mathbb{1} \to A)$$

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Ex. Module cats over H-comod ← fin. dim Hopf

$\longleftrightarrow$ H-comodule algebras

Semisimple algebras & semisimple module categories

An algebra object is <u>simple</u> if it has no subbimodule objects, semisimple if it's a finite direct product of simple algebras.

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This fails if $\mathcal{C}$ is not semisimple: $|LHS| \leq |RHS|$

Semisimple algebras & exact module categories

A module category is exact if

[projective] $\triangleright$ [anything] is [projective]

i.e. $\forall P \in \mathcal{C}\text{-proj}, X \in \mathcal{M} : P \triangleright X \in \mathcal{M}\text{-proj}$

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Conjecture [Etingof-Ostrik, 2019]

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Thm [Coulembier - S. - Zorman '25] Conjecture is true.

Ideals in categories

An ideal $\mathcal{I}$ in a category $\mathcal{A}$ consists of subspaces $\mathcal{I}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{A}}(X,Y)$ s.t.

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Ex. $\mathrm{Rad}\,\mathcal{A} \trianglelefteq \mathcal{A}$, Jacobson radical of $\mathcal{A}$

𝒞-stable Ideals in 𝒞-module categories

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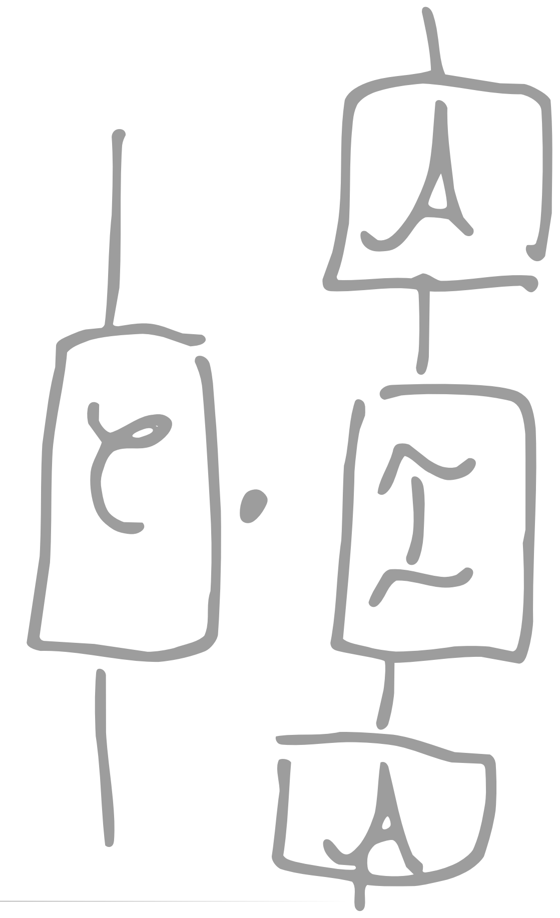
$$\mathcal{A}\,\mathcal{I}\,\mathcal{A} \subseteq \mathcal{I}, \quad \mathcal{C} \triangleright \mathcal{I} \subseteq \mathcal{I}$$



Ex. for $X \in \cancel{\mathcal{A}}\ \mathcal{M}$: $\langle \mathrm{id}_X \rangle = \{ g \mid g$ factors through $\cancel{X}$ $V \triangleright X$ for some $V \}$

Ex. $\mathcal{J}(\mathcal{M}) = \{ g \mid P \triangleright g \in \mathrm{Rad}(\mathcal{M}) \ \forall P \in \mathcal{C}\text{-proj} \}$

$\in \mathcal{M}\text{-proj}$

[$\mathcal{C}$-proj stable in $\mathcal{M}$-proj]

Even more ideals

Thm. For an algebra object A in finite $\otimes$-cat:

CSZ $$\{\text{ideals in } A\} \simeq \{(\mathcal{C}\text{-proj})\text{-stable ideals in } \mathrm{proj}_{\mathcal{C}}(A)\}$$

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Pf of $\Leftarrow$: $P_1 \xrightarrow{f \,\in \mathcal{J}(A)} P_0 \twoheadrightarrow M \rightsquigarrow \underset{\in\, \mathcal{C}\text{-proj}}{Q} \triangleright P_1 \xrightarrow{Q \triangleright f \,\in \mathrm{Rad}} Q \triangleright P_0 \to Q \triangleright M$

exactness of $\mathrm{mod}_{\mathcal{C}} A \Rightarrow Q \triangleright M$ proj $\Rightarrow Q \triangleright f = 0 \Rightarrow f = 0$ □

Etingof-Ostrik conjecture via radicals

Thm [Coulembier - S. - Zaman].

$\mathcal{J}(A)$ is the greatest nilpotent ideal in A

& TFAE

i) $\mathcal{J}(A) = 0$

ii) A is semisimple

iii) A has no non-zero nilpotent ideals

iv) $\operatorname{mod}_{\mathcal{C}} A$ is an exact $\mathcal{C}$-module category

(the other?) Gabriel's theorem

Thm [Gabriel, '68]. For $\mathbb{k}$-algebraically closed

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$$\begin{tikzcd} A \arrow[r, two heads, "\pi"] & A/J \arrow[l, bend left, "\sigma"] \arrow[d, phantom, "\cong" rotate] \\ & \Bbbk\{\text{vertices}\} \end{tikzcd} , \quad \begin{tikzcd} J \arrow[r, two heads, "p"] & J/J^2 \arrow[l, bend left, "s"] \arrow[d, phantom, "\cong" rotate] \\ & \Bbbk\{\text{arrows}\} \end{tikzcd}$$

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Good news! (& an asterisk)

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☺ Given $\mathcal{M}$, finding $S, \mathcal{E}$ s.t. $\mathcal{M} \simeq \mathrm{mod}_{\mathcal{C}}\, T_S(\mathcal{E})/I$ is "easy": $S = \underline{\mathrm{End}}\Big(\bigoplus_{L \in \mathrm{Irr}(\mathcal{M})/\mathcal{C}} L\Big)$, $\mathcal{E} = \underline{\mathrm{Ext}}^1\Big(\bigoplus_{L \in \mathrm{Irr}(\mathcal{M})/\mathcal{C}} L\Big)$

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see [Keller]

[König-Külshammer-Ovsienko]

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$$\mathrm{Rep}_{\mathcal{C}}(Q) = \left\{ \left(M_1 \in \mathcal{C}, M_2 \in \mathcal{C}, \epsilon_{12} \longrightarrow \underline{\mathcal{H}om}(M_1, M_2) \right) \right\}$$

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E.g. $\mathbb{1} \xrightarrow{\mathrm{id}_\Phi} \Phi$

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$$
\begin{tikzcd}
\epsilon_{12} \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}(M_1, M_2) \arrow[d, equal] \\
\Phi \arrow[dr] & M_1^* \otimes M_2 \arrow[d, equal] \\
& \mathbb{1} \otimes \Phi
\end{tikzcd}
$$

E.g. $\mathbb{1} \xrightarrow{id_\Phi} \Phi$

$[\mathbb{1} \xrightarrow{\mathrm{id}_\Phi} \Phi]$ $\qquad$ $[\mathbb{1} \xrightarrow{0} 0]$

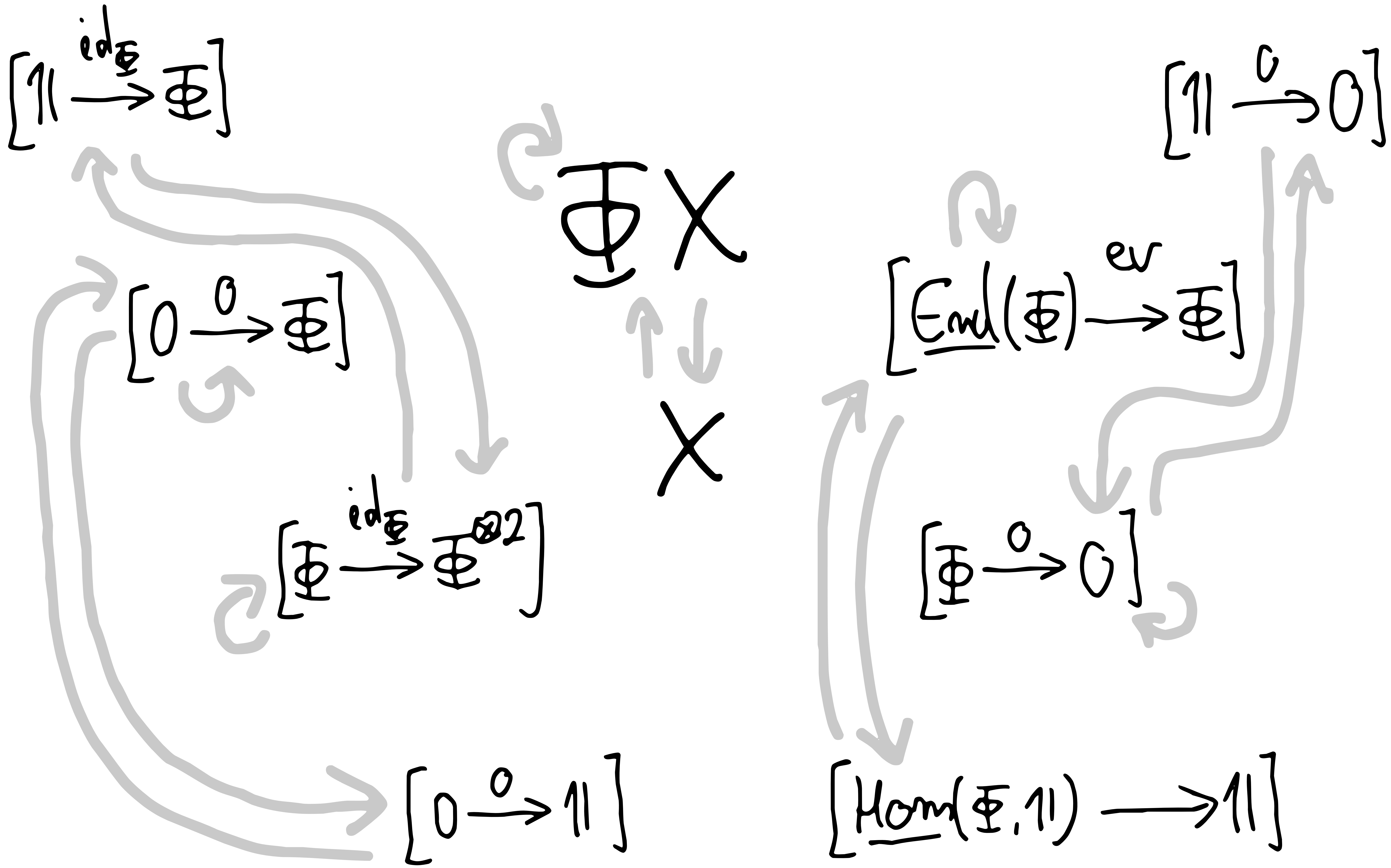
$$\Phi X$$

$[0 \xrightarrow{0} \Phi]$ $\qquad$ $[\underline{\mathrm{End}}(\Phi) \xrightarrow{\mathrm{ev}} \Phi]$

$$X$$

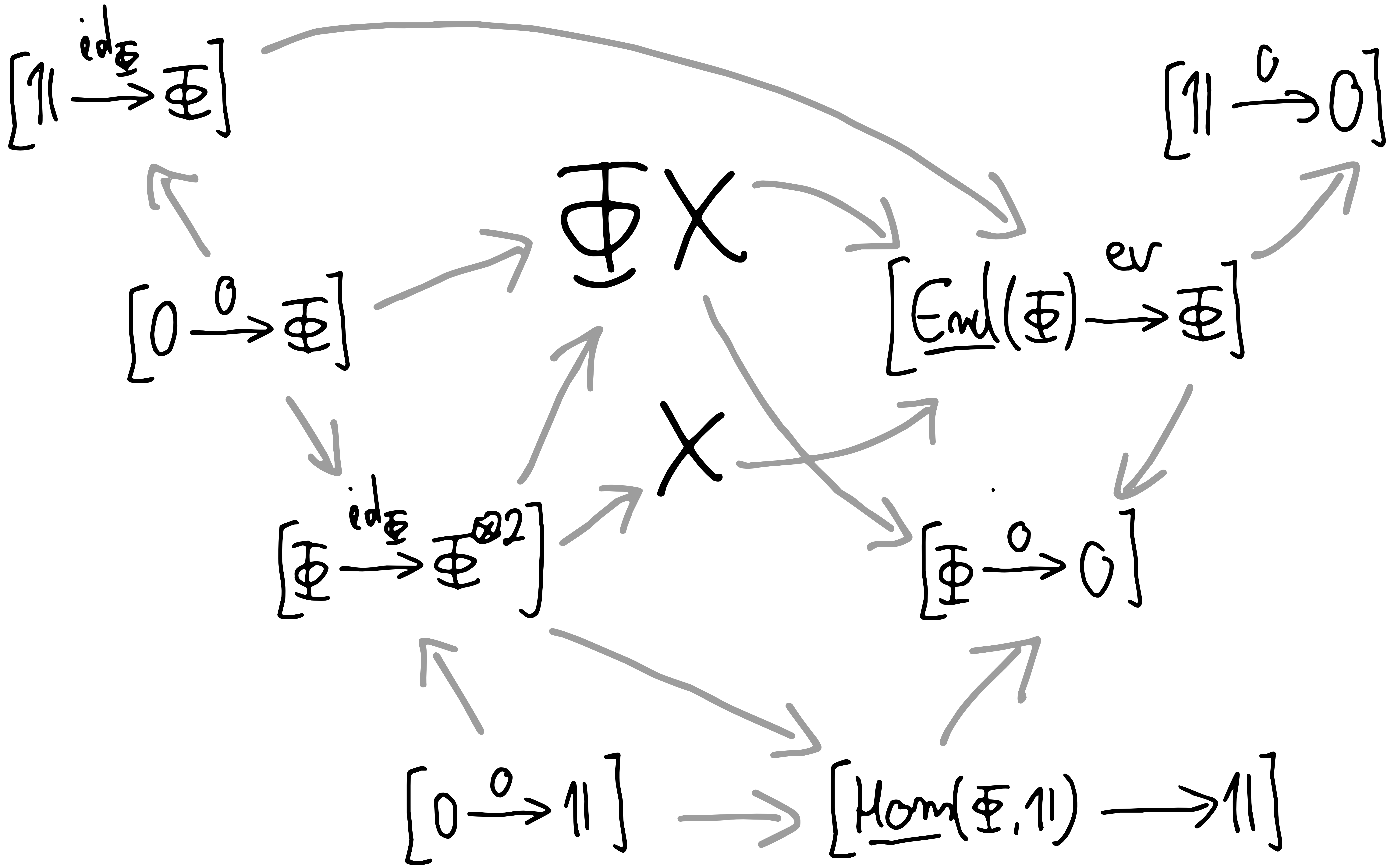
$[\Phi \xrightarrow{\mathrm{id}_\Phi} \Phi^{\otimes 2}]$ $\qquad$ $[\Phi \xrightarrow{0} 0]$

$[0 \xrightarrow{0} \mathbb{1}]$ $\qquad$ $[\underline{\mathrm{Hom}}(\Phi, \mathbb{1}) \longrightarrow \mathbb{1}]$


$[\mathbb{1} \xrightarrow{id_\Phi} \Phi]$
ΦX
X
$[\mathbb{1} \xrightarrow{0} 0]$
$[0 \xrightarrow{0} \Phi]$
$[\underline{End}(\Phi) \xrightarrow{ev} \Phi]$
$[\Phi \xrightarrow{id_\Phi} \Phi^{\otimes 2}]$
$[\Phi \xrightarrow{0} 0]$
$[0 \xrightarrow{0} \mathbb{1}]$
$[\underline{Hom}(\Phi, \mathbb{1}) \longrightarrow \mathbb{1}]$

$$
\begin{tikzcd}
{[\mathbb{1} \xrightarrow{\mathrm{id}_\Phi} \Phi]} \arrow[ddrrrr, bend left] & & & & & {[\mathbb{1} \xrightarrow{0} 0]} \\
& & & \Phi X \arrow[dr] \arrow[dddr] & & \\
& {[0 \xrightarrow{0} \Phi]} \arrow[uul] \arrow[urr] \arrow[ddr] & & & {[\underline{\mathrm{End}}(\Phi) \xrightarrow{ev} \Phi]} \arrow[uur] \arrow[dd] & \\
& & & X \arrow[ur] & & \\
& & {[\Phi \xrightarrow{\mathrm{id}_\Phi} \Phi^{\otimes 2}]} \arrow[uuur] \arrow[ur] \arrow[drr] & & {[\Phi \xrightarrow{0} 0]} & \\
& & {[0 \xrightarrow{0} \mathbb{1}]} \arrow[u] \arrow[rr] & & {[\underline{\mathrm{Hom}}(\Phi, \mathbb{1}) \longrightarrow \mathbb{1}]} \arrow[u] &
\end{tikzcd}
$$

Thank you!



$[\mathbb{1} \xrightarrow{id_\Phi} \Phi]$
$[\mathbb{1} \xrightarrow{0} 0]$
ΦX
$[0 \xrightarrow{0} \Phi]$
$[\underline{End}(\Phi) \xrightarrow{ev} \Phi]$
X
$[\Phi \xrightarrow{id_\Phi} \Phi^{\otimes 2}]$
$[\Phi \xrightarrow{0} 0]$
$[0 \xrightarrow{0} \mathbb{1}]$
$[\underline{Hom}(\Phi, \mathbb{1}) \longrightarrow \mathbb{1}]$