Zhu Algebra of Permutation Orbifold Vertex Operator Algebras

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Outline

Basics

Definition. A **vertex operator algebra***(VOA)* $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}$ -graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$ with a linear map

$$Y(\cdot,\ z) : V \to (\operatorname{End} V)[[z, z^{-1}]]$$
$$Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, v_n \in \operatorname{End} V$$

and two distinguished vectors:

- *vacuum vector* $\mathbf{1} \in V_0$
- the *Virasoro element* $\omega \in V_2$

which satisfy the following conditions:

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}; \text{For } v \in V_n, L(0)\, v = nv;$$

$+$ Jacobi Identity $+$ Some axioms.

Automorphisms of a VOA

Definition. An ***automorphism*** of a VOA $(V, Y, \mathbf{1}, \omega)$ is a linear isomorphism $g\colon V \to V$ such that

$$g(\omega) = \omega, \qquad g\,Y(u,z)\,g^{-1} = Y\big(gu, z\big) \quad (\forall\, u \in V).$$

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$$V^G := \{\, v \in V \mid g\,v = v \text{ for all } g \in G\}$$

is called the ***orbifold VOA*** of V by G.

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Example. The moonshine VOA $V^\natural$ is obtained as the orbifold

$$V^\natural = V_\Lambda^{\mathrm{Aut}(\Lambda)}$$

of the Leech-lattice VOA [Frenkel–Lepowsky–Meurman, 1984–85].

Twisted Modules in Orbifold Theory

A **key feature** of orbifold theory is the appearance of g-**twisted** V-**modules** for finite-order $g \in \mathrm{Aut}(V)$.

Let $o(g) = T$. Decompose V into g-eigenspaces:

$$V = \bigoplus_{r=0}^{T-1} V^r, \qquad gv = e^{-2\pi ir/T} v \ (v \in V^r).$$

Definition. A g-**twisted admissible module** is a $\frac{1}{T}\mathbb{Z}_{\geq 0}$-graded space

$$M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_{\geq 0}} M(n),$$

equipped with twisted vertex operators $Y_M(v,z) = \sum_{n \in \frac{r}{T}+\mathbb{Z}} v_n z^{-n-1}$, $(v \in V^r)$ whose modes satisfy the corresponding twisted Jacobi identity.

Remark. If $g = \mathrm{Id}$, this definition reduces to that of an admissible module.

Permutation Orbifolds

Let V be a VOA. Then $V^{\otimes k}$ is also a VOA. For $\sigma \in S_k$, the automorphism σ acts on $V^{\otimes k}$ by permuting the tensor factors:

$$\sigma\left(v_1 \otimes \cdots \otimes v_k\right) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

The fixed-point subalgebra $\left(V^{\otimes k}\right)^G$, where $G \leq S_k$, is called a ***permutation orbifold***.

Origins in Conformal Field Theory

Permutation orbifolds naturally arise as orbifolds of tensor product conformal field theories by symmetric group actions.

- First systematic study in physics: **[Borisov–Halpern–Schweigert, 1997]**.
    - Investigated cyclic permutation orbifolds of rational CFTs.
    - Computed twisted sectors, genus-one characters, and fusion rules.
- Generalized to arbitrary finite permutation groups by **[Bantay, 1998; 2002]**.
    - Developed explicit expressions for partition functions, modular transformations, and fusion coefficients.
    - Introduced the use of covering surfaces and Λ-matrices.

VOA Progress on Permutation Orbifolds

[Barron–Dong–Mason, 2002] Let $g = (1, 2, \cdots, k)$.

$$\{\text{irreducible } V\text{-modules}\} \underset{\longleftrightarrow}{1-1} \left\{\text{irreducible g-twisted } V^{\otimes k}\text{-modules}\right\}$$

For an irreducible V-module M, denote by $T_g(M)$ the irreducible g-twisted $V^{\otimes k}$-module associated to M.

Progress on Permutation Orbifolds

- Conformal nets approach to permutation orbifolds have been given in **[Kac–Longo–Xu, 2005]**.
- The C_2-cofiniteness of permutation orbifolds and general cyclic orbifolds was established in **[Abe, 2012, 2013; Miyamoto, 2013, 2015]**.
- Twisted modules for tensor product VOSA and permutation automorphism of even order: **[Barron-Vander Werf, 2014]**.
- Twisted modules for tensor product VOSA and permutation automorphism of odd order: **[Barron, 2016]**.

Progress on Permutation Orbifolds: Examples

- The permutation orbifolds of the lattice VOA with $k = 2$ and $k = 3$: **[Dong-Xu-Y., 2016, 2017, 2018]** and **[Bakalov-Elsinger-Kac-Todorov, 2023]**.
- S_3-Permutation orbifolds of Heisenberg VOA: **[Milas–Penn–Shao, 2019]**
- S_3-Permutation orbifolds of rank 3 Fermionic vertex superalgebras: **[Milas-Penn-Wauchope, 2019]**.
- 2-cyclic, 3-cyclic and S_3-permutation orbifolds of Virasoro VOAs **[Milas–Penn–Sadowski, 2021, 2023]**
- 2-Permutation orbifolds of W-algebras: **[Milas–Penn, 2025]**

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Progress on Permutation Orbifolds: S-matrix and Fusion Products

- S-matrix of $(V^{\otimes k})^{\langle g \rangle}$: **[Dong–Xu–Y., 2022]**.
- Fusion products involving twisted modules: **[Dong–Li–Xu–Y., 2023]; [Dong–Xu–Y., 2024]**.

Question: Zhu Algebra

How is the Zhu algebra of $(V^{\otimes k})^G$ related to that of V or $V^{\otimes k}$?

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Zhu algebra of $\mathbb{Z}_2$-permutation orbifold:

$$A\left((V \otimes V)^{\mathbb{Z}_2}\right) = ?$$

Motivations of studying $A((V\otimes V)^{\mathbb{Z}_2})$

Special status of the $\mathbb{Z}_2$ orbifold

$$(V\otimes V)^{\mathbb{Z}_2}$$

- *Smallest non-trivial permutation orbifold.*
- *[Longo–Xu; 2004]: the $\mathbb{Z}_2$-orbifold detects the rational / irrational nature of the original VOA V.*

Goal: Compute $A\big((V\otimes V)^{\mathbb{Z}_2}\big)$ to find structure and representation theory revealed by the 2-cycle permutation orbifold.

Zhu Algebra $A(V)$

[Zhu, 1996]

Let $V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n$ be a VOA. For homogeneous elements $a \in V$, define two operations:

$$a * b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(a)}}{z} Y(a,z)b,$$
$$a \circ b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b.$$

Define:

$$O(V) = \mathrm{span}\{a \circ b \mid a, b \in V\}, \quad A(V) = V/O(V).$$

$A(V)$-Theory

Theorem [Zhu, 1996]:

(1) $O(V)$ is a two-sided ideal with respect to the product $*$, so the multiplication $*$ is well-defined on $A(V) = V/O(V)$.

(2) $A(V)$ is an associative algebra with identity $[\mathbf{1}]$ and central element $[\omega]$, where $[v]$ denotes the image of $v \in V$ in $A(V)$.

(3) There is a one-to-one correspondence between **irreducible admissible** V**-modules** and **irreducible** $A(V)$**-modules**.

(4) If V is rational, then $A(V)$ is a semisimple finite-dimensional associative algebra.

Recall:

- A VOA V is **rational** if every admissible V-module is completely reducible.

$A(V)$-Theory

- Let M be an admissible V-module,
$$M = \bigoplus_{n\geq 0} M(n), \qquad M(0) \neq 0.$$
- For homogeneous $v \in V$, set $o(v) = v_{\mathrm{wt}(v)-1}$. Then $o(v)$ preserves each graded piece $M(n)$, in particular $M(0)$.
- Zhu proves that on $M(0)$ one has
$$o(u * v) = o(u)\, o(v), \qquad o(x) = 0 \text{ for all } x \in O(V),$$
hence the action factors through $A(V) = V/O(V)$ and $M(0)$ is an $A(V)$-module via $[v] \mapsto o(v)$.

- Conversely, for any $A(V)$-module U there is an admissible V-module whose top level is U; in particular irreducible admissible V-modules correspond bijectively to irreducible $A(V)$-modules.

Interpretation: $A(V)$ controls the possible top levels of admissible V-modules.

- **Theorem**. If V is C_2-cofinite, then $A(V)$ is finite-dimensional.

Recall:
V is C_2-**cofinite** if $\dim V/C_2(V) < \infty$, where $\mathrm{C}_2(V) = \mathrm{span}\{a_{-2}b \mid a, b \in V\}$.

Conjecture: Rationality and Semisimplicity of $A(V)$

Conjecture: Let V be a VOA. Then

$$\boxed{V \text{ is rational} \quad \Longleftrightarrow \quad A(V) \text{ is semisimple}}$$

Known partial results:

- If V is rational $\Rightarrow A(V)$ is semisimple. [Zhu, 1996]
- If $A(V)$ is semisimple and V is C_2-cofinite $\Rightarrow V$ is rational. [McRae, 2021]
- Without assuming C_2-cofiniteness, this conjecture remains **open**.

Higher Level Zhu Algebra $A_n(V)$

[Dong-Li-Mason, 1998]

Let n be a nonnegative integer. For homogeneous $u, v \in V$, define

$$u \circ_n v = \mathrm{Res}_z\, Y(u,z)v\, \frac{(1+z)^{\mathrm{wt}\, u+n}}{z^{2n+2}},$$

and

$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_z\, Y(u,z)v\, \frac{(1+z)^{\mathrm{wt}\, u+n}}{z^{n+m+1}}.$$

Both products extend linearly to V.

$$O_n(V) = \mathrm{span}\{\, u \circ_n v,\ (L(-1)+L(0))u \mid u, v \in V\}, \quad A_n(V) = V/O_n(V).$$

Denote the image of $v \in V$ in $A_n(V)$ by $[v]_n$.

Relation to Zhu's Algebra

When $n = 0$, this construction recovers Zhu's algebra:

$$A_0(V) = A(V), \qquad O_0(V) = O(V)$$ **[Zhu, 1996]**.

Properties of Higher Level Zhu Algebra $A_n(V)$

Theorem. **[Dong-Li-Mason, 1998]**

1. The product $*_n$ induces on $A_n(V)$ the structure of an associative algebra with identity $[1]_n$ and central element $[\omega]_n$.
2. There is a natural surjective homomorphism $A_n(V) \twoheadrightarrow A_{n-1}(V)$.
3. If $M = \bigoplus_{k\geq 0} M(k)$ is an admissible V-module with $M(0) \neq 0$, then each $M(k)$ for $k \leq n$ carries a natural structure of an $A_n(V)$-module.
4. There is a bijection between equivalence classes of simple $A_n(V)$-modules that at do not factor through $A_{n-1}(V)$ and simple admissible V-modules.
5. V is rational if and only if every $A_n(V)$ is finite-dimensional and semisimple.

Higher Level g-twisted Zhu Algebra $A_{g,n}(V)$

Recall that

$$V = \bigoplus_{r=0}^{T-1} V^r, \qquad gv = e^{-2\pi i r/T} v \ \ (v \in V^r).$$

Fix $n = l + \frac{i}{T} \in \frac{1}{T}\mathbb{Z}_{\geq 0}$ with $l \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq T-1$. Define

$$\delta_i(r) = \begin{cases} 1, & i \geq r, \\ 0, & i < r, \end{cases} \qquad \text{and} \qquad \delta_i(T) = 1.$$

For homogeneous $u \in V^r$ and any $v \in V$, set

$$u \circ_{g,n} v = \mathrm{Res}_z \; Y(u,z)v \, \frac{(1+z)^{\mathrm{wt}\, u - 1 + \delta_i(r) + l + \frac{r}{T}}}{z^{2l + \delta_i(r) + \delta_i(T-r)}}.$$

$$u *_{g,n} v = \begin{cases} \displaystyle\sum_{m=0}^{l} (-1)^m \binom{m+l}{l} \mathrm{Res}_z \; Y(u,z)v \, \frac{(1+z)^{\mathrm{wt}\, u + l}}{z^{l+m+1}}, & u \in V^0, \\ 0, & u \in V^r, \;\; r > 0. \end{cases}$$

Define

$$O_{g,n}(V) = \mathrm{span}\{u \circ_{g,n} v, L(-1)u + L(0)u\}$$

and

$$A_{g,n}(V) := V \,/\, O_{g,n}(V).$$

Basic properties of $A_{g,n}(V)$

Theore. [Dong-Li-Mason, 1998]

1. The product $*_{g,n}$ induces on $A_{g,n}(V)$ an associative algebra with identity $[1]$ and central $[\omega]$.
2. The identity map on V induces a natural surjective algebra homomorphism
$$\rho_{n,\, n-\frac{1}{T}} : \ A_{g,n}(V) \twoheadrightarrow A_{g,\, n-\frac{1}{T}}(V), \qquad [v]_n \longmapsto [v]_{n-\frac{1}{T}}.$$
3. Let $M = \bigoplus_{m\geq 0,\ m\in\frac{1}{T}\mathbb{Z}} M(m)$ be an admissible g-twisted module with $M(0) \neq 0$. Assume that M is simple. Then each $M(i)$ for $i = 0, \dots, n$ is a simple $A_{g,n}(V)$–module, and $M(i)$ and $M(j)$ are inequivalent $A_{g,n}(V)$–modules.

Basic properties of $A_{g,n}(V)$

Theorem [Dong–Li–Mason, 1998]. Assume that V is *g-rational*. Then:

- For every $n \in \frac{1}{T}\mathbb{Z}_{\geq 0}$, the algebra $A_{g,n}(V)$ is finite-dimensional and semisimple.
- The simple $A_{g,n}(V)$-modules which do *not* factor through $A_{g,n-\frac{1}{T}}(V)$ are in one-to-one correspondence with simple admissible g-twisted V-modules.
- The category of finite-dimensional $A_{g,n}(V)$-modules whose simple constituents do not factor through $A_{g,n-\frac{1}{T}}(V)$ is equivalent to the category of ordinary g-twisted V-modules.

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Theorem [Dong–Li–Mason, 1998]. V is g-rational if and only if $A_{g,n}(V)$ is finite-dimensional and semisimple for all $n \in \frac{1}{T}\mathbb{Z}_{\geq 0}$.

Definition. V is called *g-rational* if every admissible g-twisted V-module is completely reducible.

Main Results: $A_g(V^{\otimes k})$

Theorem. [Dong-Xu-Y., 2022] Let V be a VOA, and let $g = (1\ 2\ \cdots\ k) \in S_k$ be a cyclic permutation. Define the map:

$$\phi : A_g\left(V^{\otimes k}\right) \longrightarrow A\left(V\right),$$
$$\left[\sum_{a=1}^{k} u^a\right] \mapsto \left[k\Delta_k(1)u\right].$$

Then ϕ gives an isomorphism between $A_g\left(V^{\otimes k}\right)$ and $A(V)$.

Main Results

Theorem. [Dong-Xu-Y., 2025] Let V be a VOA whose ordinary modules have nonnegative conformal weights. Let $\mathcal{U} = (V \otimes V)^{\mathbb{Z}_2}$ be the 2-cycle permutation orbifold.

(1) Case $V_1 \neq 0$.

$$A(\mathcal{U}) \;\cong\; \big(A(V) \otimes A(V)\big)^{\mathbb{Z}_2} \;\oplus\; A_1(V).$$

(2) Case $V_1 = 0$.

$$A(\mathcal{U}) \;\cong\; \big(A(V) \otimes A(V)\big)^{\mathbb{Z}_2} \;\oplus\; A_1(V) \;\oplus\; \mathrm{End}\, V_3.$$

Remark The bottom-level action of $A(\mathcal{U})$ is generated by two pieces:

- *Untwisted sector*: $(A(V) \otimes A(V))^{\mathbb{Z}_2}$;
- *Twisted sector for* $\sigma = (1\ 2)$: $A_{\sigma,1/2}(V \otimes V) \cong A_1(V)$.

$V \otimes V$-**modules**	$(V \otimes V)^{\mathbb{Z}_2}$-**modules**	**Top level**
$M^i \otimes M^j, \ i \neq j$	$M^i \otimes M^j, \ i \neq j$	$M^i(0) \otimes M^j(0)$
$M^i \otimes M^i$	$(M^i \otimes M^i)^+ \oplus (\mathrm{M}^i \otimes M^i)^-$	$[(M^i(0) \otimes M^i(0)]^{\pm}$
$T_\sigma(M^i)$	$T_\sigma(M^i)^+ \oplus \mathrm{T}_\sigma(M^i)^-$	$T_\sigma(M^i)(0)^+ = M^i(0)$ $T_\sigma(M^i)(0)^- = M^i(1)$

Thank You !